

# From ADM Geometrodynamics to Ashtekar Connection Dynamics: A Compact Pedagogical Abstract-Index Derivation

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June 9, 2026

## Abstract

These notes present a systematic derivation of the Hamiltonian formulation of general relativity and Ashtekar's new variables. We begin with the Dirac–Bergmann theory of constrained Hamiltonian systems, emphasizing how singular Lagrangians give rise to primary and secondary constraints and why this structure is essential for general relativity. After establishing the required notation, including Penrose's abstract index convention and vielbein-based internal indices, we develop the  $3 + 1$  decomposition of spacetime in detail. The lapse function, shift vector, spatial metric, spatial projection, extrinsic curvature, spatial covariant derivative, and Lie evolution of spatial tensors are introduced as the geometric ingredients needed to derive the ADM Hamiltonian formulation from the Einstein–Hilbert action.

The ADM canonical variables  $(h_{ab}, \pi^{ab})$  are then derived explicitly, and the Hamiltonian and momentum constraints are shown to arise from the consistency of the primary constraints associated with lapse and shift. Finally, we reformulate the canonical theory using Ashtekar's new variables, replacing metric dynamics by connection dynamics through the canonical pair  $(A_a^i, E_i^a)$ . In this language, the gravitational constraints become the Gauss, vector, and scalar constraints, with the scalar constraint taking a polynomial form after introducing the densitized lapse. The notes conclude by emphasizing the gauge-theoretic interpretation of the Ashtekar formulation and the role of the reality conditions required to recover real Lorentzian general relativity.

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## 1 Introduction

General relativity is most naturally formulated as a covariant field theory on spacetime. Its fundamental dynamical object is the metric tensor, and its equations of motion follow from the Einstein–Hilbert action by varying the spacetime geometry itself [3]. In this Lagrangian formulation, no preferred notion of time is built into the theory: spacetime is treated as a four-dimensional geometric entity, and different choices of coordinates or foliations correspond merely to different descriptions of the same physical geometry. This feature is conceptually central to general relativity, but it also makes the transition to Hamiltonian mechanics highly nontrivial.

Hamiltonian mechanics, by contrast, requires a distinguished time parameter. To pass from a covariant spacetime theory to a canonical theory, one must first decompose spacetime into a family of spatial hypersurfaces labeled by a time parameter. This is the purpose of the  $3 + 1$  decomposition [5, 4]. Once a foliation  $\{\Sigma_t\}$  is chosen, the spacetime metric can be rewritten in terms of the lapse function  $N$ , the shift vector  $N^a$ , and the induced spatial metric  $h_{ab}$ . The lapse describes how neighboring hypersurfaces are separated in the normal direction, while the shift describes how the spatial coordinates are displaced tangentially from one hypersurface to the next. In this way, the problem of spacetime dynamics is reformulated as the evolution of spatial geometry.

However, the Hamiltonian formulation of general relativity is not simply obtained by applying the ordinary Legendre transform to the Einstein–Hilbert action. The reason is that the Lagrangian of general relativity is singular. In the  $3 + 1$  decomposition, the time derivatives of the lapse and shift do not appear as genuine dynamical velocities. Consequently, their conjugate momenta vanish, producing primary constraints. The consistency of these primary constraints under time evolution then generates secondary constraints. This is precisely the situation described by the Dirac–Bergmann theory of constrained Hamiltonian systems [1, 2].

The appearance of constraints is not a technical accident, but a direct reflection of the geometric invariance of general relativity. The lapse and shift are not ordinary dynamical fields in the same sense as the spatial metric. Instead, they encode the freedom to choose how spacetime is foliated and how spatial coordinates are transported between hypersurfaces. Their arbitrariness leads to the Hamiltonian and momentum constraints. The momentum constraint generates spatial diffeomorphisms on  $\Sigma_t$ , while the Hamiltonian constraint is associated with deformations of the hypersurface in the normal direction. Therefore, the canonical Hamiltonian of general relativity is not a conventional energy function generating evolution with respect to an absolute external time; rather, it is a sum of constraints weighted by arbitrary lapse and shift fields.

The ADM formulation makes this structure explicit. The canonical variables are the induced spatial metric  $h_{ab}$  and its conjugate momentum  $\pi^{ab}$ , which is related to the extrinsic curvature  $K_{ab}$  of the hypersurface. In these variables, the Hamiltonian of vacuum general relativity takes the constrained form

$$H[N, N^a] = \int_{\Sigma_t} d^3x \sqrt{h} (NC + N^a C_a),$$

where  $C = 0$  is the Hamiltonian constraint and  $C_a = 0$  is the momentum constraint [5, 3]. This expression reveals the deep difference between general relativity and ordinary mechanical systems: the dynamics is entirely governed by constraints. The physical phase space is not the full space of pairs  $(h_{ab}, \pi^{ab})$ , but the constraint surface modulo the gauge transformations generated by these constraints.

Despite its conceptual clarity, the ADM formulation has an important drawback. The Hamiltonian constraint written in terms of  $(h_{ab}, \pi^{ab})$  is highly nonlinear and non-polynomial. This becomes a serious obstruction when one attempts to quantize the theory canonically, because the constraint equations must be promoted to operator equations acting on quantum states. The complicated form of the ADM constraints, therefore, motivates the search for a better choice of canonical variables.

Ashtekar's new variables provide such a reformulation [6, 7, 8]. Instead of using the spatial metric directly as the configuration variable, one introduces a triad field, or more precisely a densitized triad  $E_i^a$ , which encodes the spatial metric while also carrying an internal  $SO(3)$  gauge freedom. The conjugate configuration variable is then taken to be a complex connection

$$A_a^i = \Gamma_a^i - iK_a^i,$$

where  $\Gamma_a^i$  is the spin connection compatible with the triad and  $K_a^i$  is the triadic form of the extrinsic curvature. In this formulation, general relativity is rewritten as a theory of connections rather than a theory of metrics.

The conceptual advantage of Ashtekar's variables is that gravity begins to resemble a Yang–Mills gauge theory. The pair  $(A_a^i, E_i^a)$  forms a canonical pair, and the constraints acquire a much simpler polynomial structure. In addition to the transformed scalar and vector constraints, the use of triads introduces a Gauss constraint associated with internal  $SO(3)$  rotations. Thus, the canonical formulation of gravity is recast in terms of gauge-theoretic variables: a connection, its conjugate electric-field-like momentum, and a set of first-class constraints.

The goal of these notes is to derive this chain of ideas in a systematic way. We begin with the Dirac–Bergmann algorithm for constrained Hamiltonian systems, since it provides the appropriate mathematical framework for singular Lagrangians. We then examine a simple model illustrating how missing velocities generate primary and secondary constraints. After that, we develop the  $3+1$  decomposition of spacetime, define spatial tensors, spatial covariant derivatives, extrinsic curvature, and the time evolution of spatial geometry. These ingredients are then used to derive the ADM canonical variables and the Hamiltonian formulation of the gravitational field. Finally, we introduce Ashtekar's new variables and show how the passage from metric dynamics to connection dynamics simplifies the constraint structure of general relativity.

We do not attempt to review the full development of loop quantum gravity, the Ashtekar–Barbero real connection, spin networks, holonomy-flux algebras, or the quantum Hamiltonian constraint. The purpose of these notes is more limited: to give a direct derivation of the classical transition from ADM geometrodynamics to the original complex Ashtekar connection variables.

## 2 Dirac–Bergmann Algorithm

The following discussion follows the standard Dirac–Bergmann analysis of singular Lagrangians and constrained Hamiltonian systems [1, 2]. To describe a system using Hamiltonian mechanics, we must first find the conjugate canonical momentum corresponding to the generalized coordinate. Canonical momentum is defined by differentiating the Lagrangian with respect to the generalized velocity:

$$p_i := \frac{\partial L(q^1, \dots, q^N, \dot{q}^1, \dots, \dot{q}^N)}{\partial \dot{q}^i} \quad (1)$$

It is expected that there should be a number of generalized velocities for every number of generalized coordinates, but there may not necessarily be a corresponding number of canonical momenta.

$$\begin{cases} p_1 = \frac{\partial L}{\partial \dot{q}^1} \\ \vdots \\ p_N = \frac{\partial L}{\partial \dot{q}^N} \end{cases} \quad (2)$$

The problem is that the specific form of the Lagrangian directly affects the existence of canonical momentum. To analyze this problem, we define the mapping matrix from generalized velocity to canonical momentum as  $J_{ij}$ .

$$J_{ij} := \frac{\partial p_i}{\partial \dot{q}^j} = \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} \quad (3)$$

Let's see a special case  $L = A_{ij}\dot{q}^i\dot{q}^j + B_{ij}q^i\dot{q}^j$

$$\begin{aligned} p_i &= \frac{\partial L}{\partial \dot{q}^i} \\ &= A_{jk} \left( \frac{\partial \dot{q}^j}{\partial \dot{q}^i} \dot{q}^k + \dot{q}^j \frac{\partial \dot{q}^k}{\partial \dot{q}^i} \right) + B_{jk} q^j \frac{\partial \dot{q}^k}{\partial \dot{q}^i} \\ &= 2 \times \left( \frac{1}{2} (A_{ij} + A_{ji}) \right) \dot{q}^j + B_{ji} q^j \\ &= 2A_{(ij)}\dot{q}^j + B_{ji}q^j \end{aligned} \quad (4)$$

Thus,  $J_{ij} = 2A_{(ij)}$  and we have a system of  $N$  non-homogeneous linear algebraic equations

$$J_{ij}\dot{q}^j = p_i - B_{ji}q^j, \quad i = 1, \dots, N \quad (5)$$

If  $J_{ij}$  is singular (non-invertible), then not all  $\dot{q}^j$  can be solved. Suppose that the rank of  $J_{ij}$  is  $Z$  and  $Z < N$ , we can separate  $p_i$  into the images of  $J_{ij}$  and the complement of the image in the codomain.

$$J_{ij}\dot{q}^j = p_i^{Image} - B_{ji}q^j, \quad i = 1, \dots, Z \quad (6)$$

$$J_{ij}\dot{q}^j = p_i^{others} - B_{ji}q^j = 0, \quad i = Z + 1, \dots, N \quad (7)$$

The elements not in the image give  $N - Z \equiv M$  constraints  $\phi$ , and all the constraints form a sub-manifold (a hypersurface) in the phase space.

$$\phi_i(q, p) = p_i^{others} - B_{ji}q^j = 0, \quad i = Z + 1, \dots, N \quad (8)$$

The  $\phi_i(q, p) = 0$  determined by Lagrangian  $L$  and canonical momentum  $p_i$  is called the **primary constraint**. Now, only for  $i = 1, \dots, Z$   $\dot{q}^i$  can be solved independently. In other words, the number of primary constraints corresponds to the number of generalized velocities that cannot be solved independently.

Given the existence of primary constraints, the consistency of the equations of motion must be re-examined from the perspective of the principle of least action.

$$\begin{aligned}\delta S &= \delta \int (\dot{q}^i p_i - H) dt = 0 \\ \Rightarrow \left( \dot{q}^i - \frac{\partial H}{\partial p_i} \right) \delta p_i - \left( \dot{p}_i + \frac{\partial H}{\partial q^i} \right) \delta q^i &= 0\end{aligned}\quad (9)$$

Because  $\delta q^i$  and  $\delta p_i$  are not all independent, the original Hamiltonian equation does not hold. That is,

$$\dot{q}^i - \frac{\partial H}{\partial p_i} \neq 0, \quad \dot{p}_i + \frac{\partial H}{\partial q^i} \neq 0 \quad (10)$$

Now we have  $N - Z \equiv M$  constraints, so the dimension of sub-manifold is  $2N - M$ . The principle of least action must now consider the action reaching its extreme value under  $M$  primary constraints, which results in the need for  $M$  Lagrange multipliers.

$$-\left( \dot{p}_i + \frac{\partial H}{\partial q^i} \right) dq^i + \left( \dot{q}^i - \frac{\partial H}{\partial p_i} \right) dp_i = \lambda^m \frac{\partial \phi_m}{\partial q^i} dq^i + \lambda^m \frac{\partial \phi_m}{\partial p_i} dp_i \quad (11)$$

The corrected equations of motion are as follows:

$$\dot{q}^i = \frac{\partial H}{\partial p_i} + \lambda^m \frac{\partial \phi_m}{\partial p_i} \quad (12)$$

$$\dot{p}_i = -\frac{\partial H}{\partial q^i} - \lambda^m \frac{\partial \phi_m}{\partial q^i} \quad (13)$$

The corresponding Poisson bracket correction is as follows:

$$\dot{f} = \frac{\partial f}{\partial q^i} \dot{q}^i + \frac{\partial f}{\partial p_i} \dot{p}_i = \frac{\partial f}{\partial q^i} \left( \frac{\partial H}{\partial p_i} + \lambda^m \frac{\partial \phi_m}{\partial p_i} \right) - \frac{\partial f}{\partial p_i} \left( \frac{\partial H}{\partial q^i} + \lambda^m \frac{\partial \phi_m}{\partial q^i} \right) = \{f, H\} + \{f, \phi_m\} \lambda^m \quad (14)$$

Even if  $\phi_m(q, p) = 0$ ,  $\{f, \phi_m\}$  is not necessarily zero, because  $\{f, \phi_m\}$  involves the derivative of  $\phi_m$ . This means that  $\phi_m$  is actually zero only on the constraint surface. To show that  $\phi_m$  is not zero everywhere in the phase space, we rewrite the constraint as follows:

$$\phi_m(q, p) \approx 0 \quad (15)$$

In Eq. (15), " $\approx$ " means that it holds only on the constraint surface. This convention will be used in the following discussion.

$$\begin{cases} \dot{q}^i = \{q^i, H\} + \{q^i, \phi_m\} \lambda^m \\ \dot{p}_i = \{p_i, H\} + \{p_i, \phi_m\} \lambda^m \end{cases} \quad (16)$$

Since constraints should not change with evolution, we have the following condition, which we call the consistency condition.

$$\{\phi_m, H\} + \{\phi_m, \phi_n\} \lambda^n \approx 0 \quad (17)$$

Here we establish some symbolic conventions:

$$\begin{cases} \{\phi_m, \phi_n\} \lambda^n \approx -\{\phi_m, H\} \\ \{\phi_m, \phi_n\} \equiv \Phi_{mn} \\ -\{\phi_m, H\} \equiv h_m \end{cases} \quad (18)$$

$$\Rightarrow \Phi_{mn} \lambda^n \approx h_m \quad (19)$$

If  $\det(\Phi_{mn}) \approx 0$  is satisfied, but some  $h_m \neq 0$  exist on the constraint surface, then additional constraints  $\psi_u(q, p) \approx 0$  that satisfy  $h_m \approx 0$  must be applied to ensure the consistency condition. This further constraint is called a **secondary constraint**. Dirac suggests that if a function on the constraint surface has zero Poisson brackets with all primary and secondary constraints, it is called **first-class**; otherwise, it is called **second-class**. That is to say, suppose that there are  $M$  primary constraints and  $U$  secondary constraints, the first class of function  $f$  on the constraint surface satisfies:

$$\begin{aligned} &f \text{ is first-class} \\ \Rightarrow &\begin{cases} \{f, \phi_m\} \approx 0, & m = 1, \dots, M \\ \{f, \psi_u\} \approx 0, & u = 1, \dots, U \end{cases} \end{aligned} \quad (20)$$

Since both primary and secondary constraints must maintain the consistency condition on the constraint surface, we have:

$$\begin{cases} \{\phi_m, H\} + \{\phi_m, \phi_n\} \lambda^n \approx 0, & m = 1, \dots, M \\ \{\psi_u, H\} + \{\psi_u, \phi_n\} \lambda^n \approx 0, & u = 1, \dots, U \end{cases} \quad (21)$$

From Eq. (21), it is easy to see that if the primary constraint  $\phi_m$  is a first-class function, then the Lagrange multiplier  $\lambda^n$  becomes an unconstrained free parameter. Therefore, an important conclusion is that the number of free Lagrange multipliers is equal to the number of primary constraints of the first class.

$$\begin{aligned} &\phi_n \text{ is first class, } n = 1, \dots, N \quad N \leq M \\ \Rightarrow &\begin{cases} \{\phi_m, H\} + \{\phi_m, \phi_n\} \lambda^n \approx 0, & m = 1, \dots, M \\ \{\psi_u, H\} + \{\psi_u, \phi_n\} \lambda^n \approx 0, & u = 1, \dots, U \end{cases} \\ \Rightarrow &\lambda^n \text{ is free, } n = 1, \dots, N \end{aligned} \quad (22)$$

The free Lagrange multiplier implies the existence of degrees of freedom that do not affect the reality of the physical system, namely, gauge degrees of freedom.

Similarly, if the consistency condition of a secondary constraint has a chance of not being satisfied, a new constraint  $\{\psi_u, H\} \equiv \psi'_v \approx 0$  must be given. This new constraint is also called a secondary constraint, and it must satisfy the consistency condition once again. Likewise, if the new secondary constraint is first-class, meaning it shares zero Poisson brackets with all other primary and secondary constraints, then more free Lagrange multipliers will be given. However, these gauge degrees of freedom from secondary constraints are not necessarily equivalent to the gauge degrees of freedom in gauge theory. This will be described in detail later, after introducing the Hamiltonian formulation of general relativity. Dirac once proposed the conjecture that all first-class constraints generate gauge transformations, whether primary or secondary. In general relativity, however, the interpretation of the Hamiltonian constraint is subtler: it generates normal deformations of the hypersurface and is therefore related both to gauge redundancy and to the refoliation freedom of spacetime.

## 2.1 A Useful Example of the Constraint

Both the Lagrangians of electromagnetic and gravitational fields lack some generalized velocities, which ultimately lead to the emergence of secondary constraints. Therefore, a simple example will be used here for discussion. Considering a Lagrangian  $L$  with  $N$  generalized coordinates  $(q^i)$ , only the generalized velocity corresponding to the first generalized coordinate  $(\dot{q}^1)$  is missing.

$$L = L(q^1, q^k; \dot{q}^k), \quad k = 2, \dots, N \quad (23)$$

This leads to the vanishing canonical momentum  $p_1$ , which also means that  $\dot{q}^1$  cannot be solved, thus satisfying the conditions of the primary constraints. Here we obtain a primary constraint  $\phi(q, p)$ .

$$p_1 = \frac{\partial L}{\partial \dot{q}^1} = 0 \Rightarrow \phi(q, p) \equiv p_1 = 0 \quad (24)$$

The corresponding Hamiltonian is also not a function of  $p_1$ .

$$H = \sum_{k=2}^N p_k \dot{q}^k - L \quad (25)$$

The corrected Hamiltonian equations of motion of  $\dot{q}^1$  and  $\dot{p}_1$  are as follows:

$$\begin{cases} \dot{q}^1 = \frac{\partial H}{\partial p_1} + \lambda \frac{\partial \phi}{\partial p_1} = \lambda \\ \dot{p}_1 = -\frac{\partial H}{\partial q^1} - \lambda \frac{\partial \phi}{\partial q^1} = -\frac{\partial H}{\partial q^1} \end{cases} \quad (26)$$

Since the only primary constraint  $\phi$  is a function that depends only on  $p_1$ , the other equations of motion are unaffected. Furthermore, this single constraint must satisfy the consistency condition Eq. (17), meaning that the constraint itself does not change over time. Therefore, we have the following secondary constraint:

$$\{\phi, H\} = \{p_1, H\} = \dot{p}_1 \approx 0 \Rightarrow \frac{\partial H}{\partial q^1} \approx 0 \quad (27)$$

Therefore, the secondary constraint is specified as follows and must also satisfy the consistency condition.

$$\begin{cases} \psi(q, p) \equiv \frac{\partial H}{\partial q^1} \\ 0 \approx \dot{\psi} = \{\psi, H\} + \{\psi, \phi\}\lambda \end{cases} \quad (28)$$

And we have

$$\{\psi, \phi\} = \{\psi, p_1\} = \frac{\partial \psi}{\partial q^1} = \frac{\partial^2 H}{(\partial q^1)^2} \quad (29)$$

An essential and simple special case is that  $H$  is linearly dependent on  $q^1$ , meaning that when  $\{\psi, \phi\} \approx 0$ , if  $\{\psi, H\} \approx 0$  also exists, then  $\dot{\psi} \approx 0$ , and there are no further constraints. Furthermore,  $\{\phi, \phi\} \approx 0$  and  $\{\psi, \phi\} \approx 0$  indicate that all constraints are first-class, and

the Lagrange multiplier  $\lambda$  is free, corresponding to a certain gauge symmetry. In fact, both electromagnetic and gravitational fields belong to this special case.

Since the subsequent discussion will be formulated in the language of field theory, the focus shifts from a discrete multi-particle system to fields defined on spacetime. Thus, whenever we refer to a generalized coordinate, we are referring to an entire field; the same interpretation applies to generalized velocities and canonical momenta. Likewise, when we speak of a constraint, we mean a **constraint field**. A single constraint field naturally implies that the constraint holds at infinitely many points in spacetime, and therefore it may equivalently be viewed as having infinitely many individual constraints.

### 3 Notation and Conventions

In these notes we use Penrose's abstract index notation, following the standard conventions used in modern treatments of general relativity [3]. This notation is very close in appearance to the ordinary component notation, but its meaning is different. The purpose of this section is to summarize the conventions used throughout the text.

**Abstract indices and concrete indices.** Lowercase Latin letters

$$a, b, c, \dots$$

are used as abstract tensor indices. They do not take numerical values. Instead, they indicate the tensorial type of an object. For example,

$$v^a$$

denotes a vector field, while

$$\omega_a$$

denotes a covector field. Similarly,

$$T^a_b$$

denotes a  $(1,1)$ -type tensor field, and

$$T^{ab}_c$$

denotes a  $(2,1)$ -type tensor field. The abstract indices  $a, b, c, \dots$  should therefore not be understood as component labels.

By contrast, lowercase Latin letters

$$i, j, k, \dots$$

are used as concrete indices. They can take numerical values. In the discussion of vielbeins and Ashtekar variables, these indices label the internal orthonormal frame. For a three-dimensional spatial hypersurface  $\Sigma_t$ , one usually has

$$i, j, k = 1, 2, 3.$$

For example, the dual vielbein is written as

$$e^i_a,$$

where  $a$  is an abstract covector index on the manifold, while  $i$  is a concrete internal frame index.

**Tensor type and free-index balance.** A tensor of type  $(k, l)$  carries  $k$  upper abstract indices and  $l$  lower abstract indices. An equation written in abstract index notation must have the same free abstract indices on both sides. For example,

$$u^a + v^a = w^a$$

is meaningful, while

$$u^a + v^b = w^a$$

is not a well-formed tensor equation, because the free indices are not balanced. The index name itself is not important; only the pattern of free indices matters. Thus  $u^a$  and  $u^b$  both represent the same vector field, provided the index is free and no contraction is involved.

Repeated abstract indices indicate contraction. For example,

$$\omega_a v^a$$

is a scalar, and

$$T^a{}_b v^b$$

is a vector. The repeated index must appear once upstairs and once downstairs. Repeated concrete indices are summed in the ordinary component sense. For example,

$$\delta_{ij} e^i{}_a e^j{}_b$$

means a sum over the internal frame labels  $i, j$ .

**Tensor products and order of indices.** In abstract index notation, tensor products are expressed by writing indices in the appropriate slots. For example, if  $\omega_a$  and  $\mu_a$  are covectors, then

$$\omega_a \mu_b$$

denotes the tensor product  $\omega \otimes \mu$ , while

$$\mu_a \omega_b$$

denotes  $\mu \otimes \omega$ . In general,

$$\omega_a \mu_b \neq \omega_b \mu_a,$$

because the two expressions correspond to different assignments of covectors to tensor slots. The order of tensor slots is therefore part of the tensorial information.

**Metric and raising/lowering of indices.** A metric tensor is written as  $g_{ab}$ , and its inverse is written as  $g^{ab}$ . They satisfy

$$g^{ab} g_{bc} = \delta^a{}_c.$$

The metric identifies vectors and covectors:

$$v_a = g_{ab} v^b, \quad \omega^a = g^{ab} \omega_b.$$

Although  $v^a$  and  $v_a$  are mathematically different tensor fields, they are naturally associated once a non-degenerate metric has been chosen. This convention will be used frequently.

For a spatial hypersurface  $\Sigma_t$ , the induced spatial metric is denoted by

$$h_{ab},$$

and its inverse is denoted by

$$h^{ab}.$$

The corresponding spatial index raising and lowering operations are

$$v_a = h_{ab} v^b, \quad \omega^a = h^{ab} \omega_b,$$

whenever the tensors are purely spatial.

**Coordinate components and coordinate bases.** Although abstract indices do not take numerical values, one may obtain components after choosing a basis. If  $x^\mu$  is a coordinate system, then the coordinate basis and dual basis are written as

$$\left(\frac{\partial}{\partial x^\mu}\right)^a, \quad (dx^\mu)_a.$$

For a tensor  $T^a{}_b$ , its components in this coordinate basis are defined by

$$T^\mu{}_\nu = (dx^\mu)_a T^a{}_b \left(\frac{\partial}{\partial x^\nu}\right)^b.$$

Equivalently, the tensor itself may be reconstructed as

$$T^a{}_b = T^\mu{}_\nu \left(\frac{\partial}{\partial x^\mu}\right)^a (dx^\nu)_b.$$

Here the Greek labels  $\mu, \nu, \dots$  are ordinary coordinate component indices, while  $a, b, \dots$  remain abstract tensor indices.

**Vielbein conventions.** When an orthonormal frame is introduced, we use the dual vielbein

$$e^i{}_a$$

and the vielbein

$$e^a{}_i.$$

They satisfy

$$e^i{}_a e^a{}_j = \delta^i{}_j.$$

On a purely spatial manifold  $\Sigma_t$ , one also has

$$e^i{}_a e^b{}_i = \delta^b{}_a.$$

If the spatial hypersurface is regarded as embedded in spacetime, then the same expression is understood as the spatial projector,

$$e^i{}_a e^b{}_i = h^b{}_a.$$

The spatial metric can be reconstructed from the dual vielbein as

$$h_{ab} = \delta_{ij} e^i{}_a e^j{}_b.$$

Similarly, its inverse is

$$h^{ab} = \delta^{ij} e^a{}_i e^b{}_j.$$

Thus the vielbein converts between abstract spatial indices and concrete internal frame indices. For example,

$$v^i = e^i{}_a v^a, \quad v^a = e^a{}_i v^i.$$

**Symmetrization and antisymmetrization.** Round brackets denote symmetrization with weight one:

$$T_{(ab)} := \frac{1}{2} (T_{ab} + T_{ba}).$$

Square brackets denote antisymmetrization with weight one:

$$T_{[ab]} := \frac{1}{2} (T_{ab} - T_{ba}).$$

For more indices,

$$T_{(a_1 \dots a_l)} := \frac{1}{l!} \sum_{\pi} T_{a_{\pi(1)} \dots a_{\pi(l)}},$$

and

$$T_{[a_1 \dots a_l]} := \frac{1}{l!} \sum_{\pi} \text{sgn}(\pi) T_{a_{\pi(1)} \dots a_{\pi(l)}},$$

where the sum runs over all permutations of  $1, \dots, l$ .

A vertical bar inside brackets means that the enclosed index is excluded from the symmetrization or antisymmetrization. For example,

$$T_{[a|c|b]} = \frac{1}{2} (T_{acb} - T_{bca}).$$

This convention is useful when writing expressions such as the torsion-free Cartan equation,

$$T^i{}_{ab} = 2\partial_{[a} e^i{}_{b]} + 2\omega^i{}_{[a|j|} e^j{}_{b]}.$$

Here the index  $j$  is not antisymmetrized; only the abstract indices  $a, b$  are.

**Useful algebraic rules for brackets.** The bracket notation is not only a definition of symmetric and antisymmetric parts, but also a useful computational device. We summarize several identities that will be used repeatedly.

First, brackets are inherited through contractions. For example, if  $S^{a_1 \dots a_l}$  is totally antisymmetric, then only the totally antisymmetric part of  $T_{a_1 \dots a_l}$  contributes to the contraction:

$$T_{a_1 \dots a_l} S^{[a_1 \dots a_l]} = T_{[a_1 \dots a_l]} S^{a_1 \dots a_l} = T_{[a_1 \dots a_l]} S^{[a_1 \dots a_l]}.$$

Similarly, if  $S^{a_1 \dots a_l}$  is totally symmetric, then

$$T_{a_1 \dots a_l} S^{(a_1 \dots a_l)} = T_{(a_1 \dots a_l)} S^{a_1 \dots a_l} = T_{(a_1 \dots a_l)} S^{(a_1 \dots a_l)}.$$

In words, when a tensor is contracted with a totally antisymmetric tensor, the symmetric part of the other tensor is irrelevant; when it is contracted with a totally symmetric tensor, the antisymmetric part is irrelevant.

Second, nested brackets of the same type may be removed. For example,

$$T_{[[ab]c]} = T_{[abc]},$$

where

$$T_{[[ab]c]} := \frac{1}{2} (T_{[abc]} - T_{[bac]}).$$

Similarly, nested round brackets may be removed:

$$T_{((ab)c)} = T_{(abc)}.$$

Thus repeated symmetrization or antisymmetrization over the same set of indices does not produce a new operation.

Third, symmetrizing and antisymmetrizing over the same set of indices gives zero. For example,

$$T_{[(ab)c]} = 0, \quad T_{(a[bcd])} = 0.$$

The reason is that a tensor cannot be simultaneously symmetric and antisymmetric in the same pair of indices unless that part vanishes.

Fourth, the contraction between a totally symmetric tensor and a totally antisymmetric tensor over the same indices vanishes:

$$T^{(abc)}S_{[abc]} = 0.$$

More generally,

$$T^{(a_1 \dots a_l)}S_{[a_1 \dots a_l]} = 0.$$

This identity is frequently useful when simplifying curvature and torsion expressions.

Finally, if a tensor is already totally symmetric, then its totally antisymmetric part vanishes:

$$T_{a_1 \dots a_l} = T_{(a_1 \dots a_l)} \implies T_{[a_1 \dots a_l]} = 0.$$

Conversely, if a tensor is already totally antisymmetric, then its totally symmetric part vanishes:

$$T_{a_1 \dots a_l} = T_{[a_1 \dots a_l]} \implies T_{(a_1 \dots a_l)} = 0.$$

For instance,

$$T_{ab} = T_{(ab)} \implies T_{[ab]} = 0,$$

and

$$T_{ab} = T_{[ab]} \implies T_{(ab)} = 0.$$

These rules will be used freely in later calculations. For example, the symmetry of the extrinsic curvature,

$$K_{ab} = K_{(ab)},$$

immediately implies

$$K_{[ab]} = 0.$$

Likewise, for the exterior product of two one-forms,

$$(\alpha \wedge \beta)_{ab} = 2\alpha_{[a}\beta_{b]},$$

only the antisymmetric part in the abstract indices  $a, b$  is relevant.

**Exterior products.** For one-forms  $\alpha_a$  and  $\beta_a$ , the exterior product is

$$(\alpha \wedge \beta)_{ab} = 2\alpha_{[a}\beta_{b]} = \alpha_a\beta_b - \alpha_b\beta_a.$$

More generally, exterior products are always antisymmetrized over the relevant covariant indices. This convention is consistent with the Cartan structure equation

$$T_{ab}^i = d_a e_b^i + \omega_a^i{}_j \wedge e_b^j.$$

**Covariant derivative and spin connection.** The spacetime covariant derivative is denoted by

$$\nabla_a.$$

After a  $3 + 1$  decomposition, the spatial covariant derivative compatible with  $h_{ab}$  is denoted by

$$D_a.$$

When a vielbein is used, the spin connection is written as

$$\omega_a^i{}_j.$$

It carries one abstract covector index  $a$  and two concrete internal indices  $i, j$ . Metric compatibility in the internal orthonormal frame implies

$$\omega_a^i{}_j = -\omega_a^j{}_i.$$

Equivalently, the spin connection is valued in the Lie algebra of the internal rotation group. In two dimensions it is  $so(2)$ -valued, while for the spatial triad in Ashtekar's formulation it is  $so(3)$ -valued.

The vielbein compatibility condition is written as

$$D_a e^i{}_b = 0,$$

or more explicitly,

$$\partial_a e^i{}_b + \omega^i{}_{aj} e^j{}_b - \Gamma^c{}_{ab} e^i{}_c = 0.$$

This equation determines the spin connection associated with a given vielbein and spatial metric. It is the same geometric object that later enters Ashtekar's connection.

**Summary of index meanings.** Throughout these notes, the following convention will be used:

| Index type              | Meaning                         | Example                          |
|-------------------------|---------------------------------|----------------------------------|
| $a, b, c, \dots$        | abstract tensor indices         | $v^a, \omega_a, h_{ab}$          |
| $i, j, k, \dots$        | concrete internal frame indices | $e^i{}_a, E^a{}_i, \Gamma^i{}_a$ |
| $\mu, \nu, \rho, \dots$ | coordinate component indices    | $T^\mu{}_\nu, (dx^\mu)_a$        |

The abstract indices encode tensorial type and contraction structure, while the concrete indices label components in a chosen basis or internal frame.

## 4 3+1 Decomposition

The geometric construction used in this and the following sections follows the standard 3 + 1 decomposition of spacetime into spacelike hypersurfaces, with lapse, shift, induced spatial metric, spatial projection, and extrinsic curvature [5, 4, 3].

Relativity broke the absolute concept of simultaneity and recognized that only spacetime is an absolute physical reality, independent of human factors. The concepts of space and time only acquire meaning after selecting a frame of reference and performing a "3+1 decomposition" of spacetime and are therefore relative. Since the most significant difference between Hamiltonian mechanics and Lagrangian mechanics is that Hamiltonian mechanics places time in a special position, it is necessary first to define what time is artificially, which inevitably involves a particular 3+1 decomposition of spacetime.

First, we can artificially define a set of spacelike hypersurfaces  $\{\Sigma_t\}$  that can be described by a specific parameter "t" in spacetime. Then, we define a timelike vector field  $t^a$  pointing to the future, and it has the relationship  $t^a \nabla_a t = 1$  with the parameter  $t$  we just used to describe the hypersurface. The vector field  $t^a$  can be used to generate world lines and therefore can be understood as a specification of the observer. All observers, combined with spacelike hypersurfaces  $\{\Sigma_t\}$ , constitute a 3+1 decomposition of spacetime. We can then decompose  $t^a$  into components that are orthogonal and tangent to  $\Sigma_t$ . The orthogonal direction to  $\Sigma_t$ , which is the unit normal vector field of spacelike hypersurfaces, is denoted as  $n^a$ , and the component in this direction is labeled with a scalar field  $N$ , usually called the lapse function. The vector field tangent to  $\Sigma_t$  is denoted as  $N^a$  and is called the shift vector field.

$$t^a = N n^a + N^a \tag{30}$$

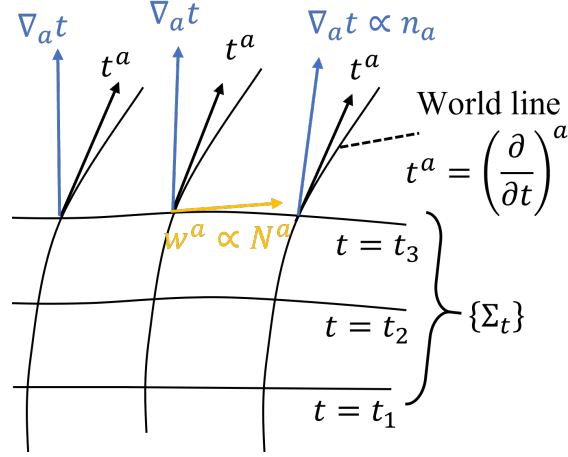


Figure 1: Schematic diagram of 3+1 decomposition of spacetime.

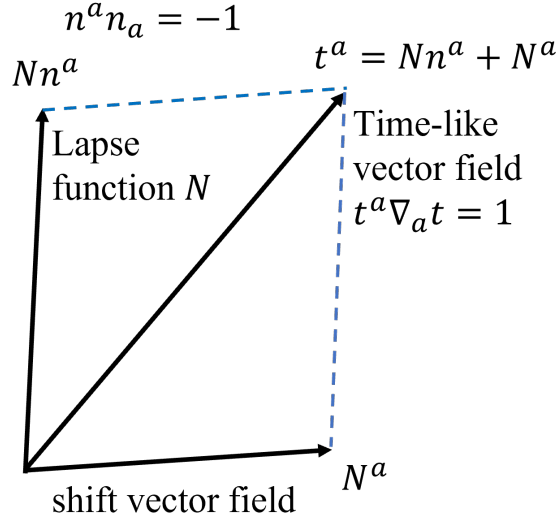


Figure 2: Schematic diagram of lapse function and shift vector field.

Since  $n^a$  is the unit timelike vector field, it naturally satisfies  $n^a n_a = -1$  everywhere. From this, we can obtain the following relationship between  $n_a$  and  $\nabla_a t$ .

$$\begin{cases} t^a \nabla_a t = 1 \\ t^a n_a = (Nn^a + N^a) n_a \end{cases} \Rightarrow \begin{cases} t^a \nabla_a t = 1 \\ t^a n_a = Nn^a n_a = -N \end{cases} \Rightarrow n_a = -N \nabla_a t \quad (31)$$

From Eq. (31), we can see that the lapse function  $N$  only depends on how  $\Sigma_t$  is layered. Now, let's introduce a coordinate system that conforms to the properties of the reference frame described above. Let  $\{x^i\}$  be a local coordinate system on  $\Sigma_0$ . The observer's world line carries this coordinate system outward, meaning that points on the world line have the same local coordinate system  $\{x^i\}$ . Then, using the parameter  $t$  labeled  $\Sigma_t$  as time, it becomes a four-dimensional coordinate system  $\{x^0 \equiv t, x^i\}$ , sometimes called the co-moving coordinate system. In such a coordinate system, we want to write down the components of metric.

$$ds^2 = g_{00} dt^2 + 2g_{0i} dt dx^i + g_{ij} dx^i dx^j \quad (32)$$

Since  $1 = (\partial/\partial t)^a (dt)_a = (\partial/\partial t)^a \nabla_a t$ , we can specify  $t^a = (\partial/\partial t)^a$ . According to Eq. (30), the  $g_{00}$  component is:

$$\begin{aligned}
g_{00} &= g_{ab} \left( \frac{\partial}{\partial t} \right)^a \left( \frac{\partial}{\partial t} \right)^b \\
&= g_{ab} (N n^a + N^a) (N n^b + N^b) \\
&= N^2 g_{ab} n^a n^b + g_{ab} N^a N^b \\
&= -N^2 + N^i N_i
\end{aligned} \tag{33}$$

Here  $N^i \equiv N^a (dx^i)_a$ ,  $N_a \equiv g_{ab} N^b$ , and  $N_i \equiv N_a (\partial/\partial x^i)^a$ . Similarly, the  $g_{0i}$  components can be written as:

$$\begin{aligned}
g_{0i} &= g_{ab} \left( \frac{\partial}{\partial t} \right)^a \left( \frac{\partial}{\partial x^i} \right)^b \\
&= g_{ab} (N n^a + N^a) \left( \frac{\partial}{\partial x^i} \right)^b \\
&= N_b \left( \frac{\partial}{\partial x^i} \right)^b \\
&= N_i
\end{aligned} \tag{34}$$

Therefore, according to Eq. (32), Eq. (33), and Eq. (34), we can write a small spacetime interval as:

$$ds^2 = (-N^2 + N^i N_i) dt^2 + 2N_i dt dx^i + g_{ij} dx^i dx^j \tag{35}$$

A more common equivalent is as follows:

$$ds^2 = -N^2 dt^2 + g_{ij} (N^i dt + dx^i) (N^j dt + dx^j) \tag{36}$$

## 5 Spatial Tensor

After performing a 3+1 decomposition on spacetime, we can meaningfully discuss the concepts of space and time. Geometric objects in spatial directions will play an essential role in the subsequent discussion because they are related to the determination of generalized coordinates.

First, based on the 3+1 decomposition, we can define a spatial projection matrix  $h_b^a$  that satisfies  $h_b^a n^b = 0$  and  $h_b^a n_a = 0$  as follows:

$$h_b^a \equiv \delta_b^a + n^a n_b \tag{37}$$

We can now define a tensor  $T_b^a$  as spatial if it satisfies the following condition.

$$\begin{cases} T_b^a n^b = 0 \\ T_b^a n_a = 0 \\ T_b^a = h_c^a h_b^d T_d^c \end{cases} \tag{38}$$

In fact, it is highly non-trivial for a tensor to satisfy spatial conditions; exceptions can occur even starting from the simplest coordinate basis. Let  $\{t, x^i\}$  be the coordinate system corresponding to the 3+1 decomposition. Then  $(\partial/\partial x^i)^a$  must be the spatial tensor; however,  $(dx^i)_a$  is not necessarily the spatial tensor.

$$\begin{aligned}
h_a^b (dx^i)_b &= \left( \delta_a^b + n_a n^b \right) (dx^i)_b \\
&= (dx^i)_a + (-N \nabla_a t) N^{-1} (t^b - N^b) (dx^i)_b \\
&= (dx^i)_a - \left( (\nabla_a t) t^b (dx^i)_b - (\nabla_a t) N^b (dx^i)_b \right) \\
&= (dx^i)_a - ((\nabla_a t) 0 - (dt)_a N^i) \\
&= (dx^i)_a + N^i (dt)_a \neq (dx^i)_a
\end{aligned} \tag{39}$$

We will see various manifestations of the nontriviality of spatial tensors later. For example, after covariant differentiation of a spatial tensor, it is usually no longer a spatial tensor. Therefore, the projection matrix  $h_b^a$  will be used frequently.

In Hamiltonian mechanics discussions of gravitational fields, two spatial tensors play crucial roles: the induced spatial metric  $h_{ab}$  and the extrinsic curvature  $K_{ab}$ . The definition of the induced spatial metric is as follows:

$$h_{ab} \equiv g_{ab} + n_a n_b \tag{40}$$

$$h^{ab} \equiv g^{ab} + n^a n^b \tag{41}$$

The definition of extrinsic curvature  $K_{ab}$  is as follows:

$$K_{ab} := h_a^c h_b^d \nabla_c n_d \tag{42}$$

Furthermore, the definition of extrinsic curvature can be simplified for the following reasons.

$$\begin{aligned}
K_{ab} &= h_a^c h_b^d \nabla_c n_d \\
&= h_a^c (\delta_b^d + n_b n^d) \nabla_c n_d \\
&= h_a^c (\nabla_c n_b + \frac{1}{2} n_b \nabla_c (n^d n_d)) \\
&= h_a^c (\nabla_c n_b + \frac{1}{2} n_b \nabla_c (-1)) = h_a^c \nabla_c n_b
\end{aligned} \tag{43}$$

Since  $n^a$  is chosen to always be orthogonal to the hypersurface  $\Sigma_t$  described by  $t$ , this ensures that  $K_{ab}$  is always symmetric.

$$K_{ab} = K_{ba} \tag{44}$$

To simplify the following discussion, the following two conventions are introduced here.

$$K := h^{ab} K_{ab} \tag{45}$$

$$h := \det(h_{ab}) \tag{46}$$

$$g := \det(g_{ab}) \tag{47}$$

## 6 Spatial Derivative of Spatial Tensor

As mentioned earlier, a spatial tensor generally ceases to be a spatial tensor after covariant differentiation. Therefore, it is necessary to define a spatial derivative  $D_a$  to perform covariant differentiation while preserving the spatial tensor. Its definition is as follows:

$$D_c T_b^a := h_a^d h_b^e h_c^f \nabla_f T_e^d \quad (48)$$

Next, we verify whether the spatial derivative  $D_a$  is compatible with the spatial metric under this definition, that is, we verify whether the inner product in the spatial part is invariant under this definition.

$$\begin{aligned} D_c h_{ab} &= h_a^d h_b^e h_c^f \nabla_f h_{de} \\ &= h_a^d h_b^e h_c^f \nabla_f (g_{de} + n_d n_e) \\ &= h_a^d h_b^e (h_c^f \nabla_f g_{de} + h_c^f \nabla_f (n_d n_e)) \\ &= h_a^d h_b^e (0 + h_c^f \nabla_f (n_d n_e)) \\ &= h_a^d h_b^e h_c^f \nabla_f (n_d n_e) \\ &= (h_a^d n_d) h_b^e h_c^f \nabla_f (n_e) + h_a^d (n_e h_b^e) h_c^f \nabla_f (n_d) = 0 + 0 \end{aligned} \quad (49)$$

Therefore, the spatial covariant derivative  $D_c$  is indeed compatible with the spatial metric  $h_{ab}$ . Due to the non-commutativity of the spatial covariant derivative, a three-dimensional spatial Riemann curvature tensor  ${}^3R_{abc}{}^d$  can be defined as:

$${}^3R_{abc}{}^d \omega_d = 2D_{[a} D_{b]} \omega_c \quad (50)$$

Since the projection matrix also requires differentiation, the calculation of geometric objects involving higher-order differentiation becomes more complex. Usually, the projection matrix is no longer used to project the four-dimensional target directly. It is worth noting that extrinsic curvature often appears as a correction term.

$$\begin{aligned} 2D_a D_b \omega_c &= 2h_c^g h_a^e h_b^m \nabla_e (h_g^f h_m^d \nabla_d \omega_f) \\ &= 2h_c^g h_a^e h_b^m \nabla_e ((\delta_g^f + n_g n^f) h_m^d \nabla_d \omega_f) \\ &= 2h_c^g h_a^e h_b^m \nabla_e (h_m^d \nabla_d \omega_g - n_g \omega_f h_m^d \nabla_d n^f) \\ &= 2h_c^g h_a^e h_b^m \nabla_e (h_m^d \nabla_d \omega_g - n_g \omega_f K_m^f) \\ &= 2h_c^g h_b^m (h_a^e \nabla_e (h_m^d \nabla_d \omega_g) - (h_a^e \nabla_e n_g) \omega_f K_m^f - h_a^e n_g \nabla_e (\omega_f K_m^f)) \\ &= 2(h_c^g h_a^e h_b^m \nabla_e (h_m^d \nabla_d \omega_g) - h_c^g K_{ag} \omega_f K_b^f - h_b^m h_a^e (h_c^g n_g) \nabla_e (\omega_f K_m^f)) \\ &= 2(h_c^g h_a^e h_b^m \nabla_e (h_m^d \nabla_d \omega_g) - K_{ac} \omega_f K_b^f - 0) \\ &= 2h_c^g h_a^e h_b^m \nabla_e \nabla_d \omega_g + 2h_c^g h_a^e h_b^m (\nabla_e h_m^d) (\nabla_d \omega_g) - 2K_{ac} K_b^f \omega_f \\ &= 2h_c^g h_a^e h_b^m \nabla_e \nabla_d \omega_g + 2h_c^g h_b^m h_a^e \nabla_e (n_m n^d) (\nabla_d \omega_g) - 2K_{ac} K_b^f \omega_f \\ &= 2h_c^g h_a^e h_b^m \nabla_e \nabla_d \omega_g + 2h_c^g (K_{ab} n^d + h_b^m n_m h_a^e \nabla_e n^d) (\nabla_d \omega_g) - 2K_{ac} K_b^f \omega_f \\ &= 2h_c^g h_a^e h_b^m \nabla_e \nabla_d \omega_g + 2h_c^g (K_{ab} n^d + 0) (\nabla_d \omega_g) - 2K_{ac} K_b^f \omega_f \\ &= 2h_a^e h_b^d h_c^g \nabla_e \nabla_d \omega_g - 2K_{ac} K_b^f \omega_f + 2K_{ab} h_c^g n^d \nabla_d \omega_g \end{aligned} \quad (51)$$

$$\begin{aligned}
2D_{[a}D_{b]}\omega_c &= 2h_{[a}^e h_{b]}^d h_c^g \nabla_e \nabla_d \omega_g - 2K_{c[a} K_{b]}^f \omega_f + 2K_{[ab]} h_c^g n^d \nabla_d \omega_g \\
&= 2h_{[a}^e h_{b]}^d h_c^g \nabla_e \nabla_d \omega_g - 2K_{c[a} K_{b]}^f \omega_f \\
&= 2h_a^{[e} h_b^{d]} h_c^g \nabla_e \nabla_d \omega_g - 2K_{c[a} K_{b]}^f \omega_f \\
&= 2h_a^e h_b^d h_c^g \nabla_{[e} \nabla_{d]} \omega_g - 2K_{c[a} K_{b]}^f \omega_f \\
&= h_a^e h_b^d h_c^g R_{edg}^m \omega_m - 2K_{c[a} K_{b]}^f \omega_f \\
&= h_a^e h_b^d h_c^g h_m^f R_{edg}^m \omega_f - 2K_{c[a} K_{b]}^f \omega_f
\end{aligned} \tag{52}$$

After the above complex calculations, we successfully obtained the following expression for the spatial Riemann curvature tensor, which is important for subsequently writing the gravitational field theory in the form of Hamiltonian mechanics.

$${}^3R_{abc}^f = h_a^e h_b^d h_c^g h_m^f R_{edg}^m - 2K_{c[a} K_{b]}^f \tag{53}$$

Next, we calculate the spatial Ricci scalar  ${}^3R$  corresponding to the spatial Riemann curvature tensor.

$$\begin{aligned}
{}^3R &= h^{ac} h^{bf} {}^3R_{abcf} \\
&= h^{ac} h^{bf} h_a^e h_b^d h_c^g h_f^m R_{edgm} - h^{ac} h^{bf} 2K_{c[a} K_{b]}^f \\
&= h^{eg} h^{dm} R_{edgm} - h^{ac} h^{bf} (K_{ca} K_{bf} - K_{cb} K_{af}) \\
&= h^{eg} h^{dm} R_{edgm} - K^2 + K_{cb} K^{cb}
\end{aligned} \tag{54}$$

The first term of Eq.(54) needs further simplification because we want to find a clear relationship between the spatial Ricci scalar  ${}^3R$  and the original four-dimensional Ricci scalar  $R$ .

$$\begin{aligned}
h^{eg} h^{dm} R_{edgm} &= (g^{eg} + n^e n^g)(g^{dm} + n^d n^m) R_{edgm} \\
&= (R + 2n^e n^g R_{eg} + n^e n^d n^g n^m R_{edgm}) \\
&= (R + 2n^e n^g R_{eg} + n^e n^d n^g n^m R_{[ed]gm}) \\
&= (R + 2n^e n^g R_{eg} + n^{[e} n^{d]} n^g n^m R_{edgm}) \\
&= R + 2n^e n^g R_{eg}
\end{aligned} \tag{55}$$

Combining Eq. (54) and Eq. (55), we obtain the following results.

$$\begin{aligned}
R &= h^{eg} h^{dm} R_{edgm} - 2n^e n^g R_{eg} \\
&= 2n^e n^g G_{eg} - 2n^e n^g R_{eg} \\
&= ({}^3R + K^2 - K_{cb} K^{cb}) - 2n^e n^g R_{eg}
\end{aligned} \tag{56}$$

We are also curious about the relationship between the spatial Ricci scalar  ${}^3R$  and the Einstein tensor  $G_{ab}$ . So we can rewrite the first term of Eq.(54) again using the following Einstein tensor.

$$G_{eg} \equiv R_{eg} - \frac{1}{2} R g_{eg} \tag{57}$$

$$\begin{aligned}
h^{eg}h^{dm}R_{edgm} &= R + 2n^en^gR_{eg} \\
&= R + 2n^en^g\left(G_{eg} + \frac{1}{2}Rg_{eg}\right) \\
&= R + 2n^en^gG_{eg} + n^en_eR \\
&= R + 2n^en^gG_{eg} - R \\
&= 2n^en^gG_{eg}
\end{aligned} \tag{58}$$

Combining Eq. (54) and Eq. (58), we obtain the following results.

$$\begin{aligned}
{}^3R &= h^{eg}h^{dm}R_{edgm} - K^2 + K_{cb}K^{cb} \\
&= 2n^en^gG_{eg} - K^2 + K_{cb}K^{cb} \\
\Rightarrow 2n^en^gG_{eg} &= {}^3R + K^2 - K_{cb}K^{cb}
\end{aligned} \tag{59}$$

Combining Eq. (59) and Eq. (57), we can again obtain the result of Eq. (56).

## 7 Time Derivative of Spatial Tensor

In the context of the 3+1 decomposition of spacetime, an important objective is to discuss the problem of time evolution, that is, to find the derivative along a vector field  $t^a$ . Finding the derivative along a vector field is called the Lie derivative. First, the Lie derivative for a scalar function  $f$  is defined as follows:

$$\begin{aligned}
(\mathcal{L}_{\vec{t}}f)|_q &:= \lim_{\Delta t \rightarrow 0} \frac{(\phi_{\Delta t}^*f - f)|_q}{\Delta t} \\
&= \lim_{\Delta t \rightarrow 0} \frac{f(\phi_{\Delta t}(q)) - f(q)}{\Delta t} \\
&= (t^a\nabla_a f)|_q
\end{aligned} \tag{60}$$

It can be observed that the Lie derivative for a scalar function  $f$  is consistent with the covariant derivative, and the definition of the Lie derivative for a tensor is the same as below, but the specific implementation will be shown later.

$$\mathcal{L}_{\vec{t}}T^a{}_b := \lim_{\Delta t \rightarrow 0} \frac{(\phi_{\Delta t}^*T^a{}_b) - T^a{}_b}{\Delta t} \tag{61}$$

The Lie derivative rule for a vector field is as follows:

$$\begin{aligned}
\mathcal{L}_{\vec{t}}u^a &= \lim_{\Delta t \rightarrow 0} \frac{(\phi_{\Delta t}^*u^a) - u^a}{\Delta t} \\
&= \lim_{\Delta t \rightarrow 0} \frac{e^{(\Delta t)\vec{t}}u^ae^{(-\Delta t)\vec{t}} - u^a}{\Delta t} \\
&= [\vec{t}, u^a] = t^b\nabla_b u^a - u^b\nabla_b t^a
\end{aligned} \tag{62}$$

Combining Eq. (60) and Eq. (62), we can obtain the following Lie derivative of the dual vector field.

$$\begin{aligned}
& \begin{cases} \mathcal{L}_{\vec{t}}(u^a w_a) = t^a \nabla_a (u^a w_a) \\ \mathcal{L}_{\vec{t}} u^a = t^b \nabla_b u^a - u^b \nabla_b t^a \end{cases} \\
& \Rightarrow \mathcal{L}_{\vec{t}} w_a = t^b \nabla_b w_a + w_b \nabla_a t^b
\end{aligned} \tag{63}$$

Based on Eq. (62) and Eq. (63), the specific implementation of the Lie derivative for a general tensor in Eq. (61) is as follows:

$$\mathcal{L}_{\vec{t}} T^a_b = t^c \nabla_c T^a_b - T^c_b \nabla_c t^a + T^a_c \nabla_b t^c \tag{64}$$

Thus, given a spatial tensor  $T^a_b$ , the time derivative can be defined as:

$$\dot{T}^a_b := h^a_c h_b^d \mathcal{L}_{\vec{t}} T^c_d \equiv \tilde{\mathcal{L}}_{\vec{t}} T^c_d \tag{65}$$

The Lie derivative along the vector field  $t^a$  can be described by two parts: the lapse function  $N$  and the shift vector field  $N^a$ , based on the 3+1 decomposition.

$$\dot{T}^c_d = \tilde{\mathcal{L}}_{\vec{t}} T^c_d = \tilde{\mathcal{L}}_{N\vec{n} + \vec{N}} T^c_d = (N\tilde{\mathcal{L}}_{\vec{n}} + \tilde{\mathcal{L}}_{\vec{N}}) T^c_d \tag{66}$$

Now, how the induced spatial metric field  $h_{ab}$  evolves is important. According to Eq. (66), we can calculate the contributions of the directions of  $\tilde{\mathcal{L}}_{\vec{n}} h_{ab}$  and  $\tilde{\mathcal{L}}_{\vec{N}} h_{ab}$  separately. Note that, to maintain a spatial tensor, we first compute the Lie derivatives and then project. First, we consider  $\mathcal{L}_{\vec{n}} h_{ab}$ .

$$\begin{aligned}
\mathcal{L}_{\vec{n}} h_{ab} &= n^c \nabla_c h_{ab} + h_{ac} \nabla_b n^c + h_{cb} \nabla_a n^c \\
&= (0 + n^c \nabla_c (n_a n_b)) + (g_{ac} \nabla_b n^c + 0) + (g_{cb} \nabla_a n^c + 0) \\
&= n_a n^c \nabla_c n_b + n_b n^c \nabla_c n_a + \nabla_b n_a + \nabla_a n_b
\end{aligned} \tag{67}$$

Then, project it to the spatial part and use the extrinsic curvature defined in Eq. (42).

$$\begin{aligned}
\tilde{\mathcal{L}}_{\vec{n}} h_{ab} &= h_a^c h_b^d \mathcal{L}_{\vec{n}} h_{cd} \\
&= 0 + 0 + h_a^c h_b^d (\nabla_d n_c + \nabla_c n_d) \\
&= K_{ba} + K_{ab} = 2K_{ab}
\end{aligned} \tag{68}$$

Perform the same operation in the shift vector direction and use the spatial covariant differentiation defined in Eq. (48).

$$\begin{aligned}
\tilde{\mathcal{L}}_{\vec{N}} h_{ab} &= h_a^c h_b^d \mathcal{L}_{\vec{N}} h_{cd} \\
&= h_a^c h_b^d (N^e \nabla_e h_{cd} + h_{ce} \nabla_d N^e + h_{ed} \nabla_c N^e) \\
&= (0 + 0) + (h_a^c h_b^d \nabla_d N_c + 0) + (h_a^c h_b^d \nabla_c N_d + 0) \\
&= D_b N_a + D_a N_b = 2D_{(a} N_{b)}
\end{aligned} \tag{69}$$

Finally, we obtain the following expression for the evolution of the spatial metric  $\dot{h}_{ab}$ , which is crucial for the subsequent calculation of canonical momentum.

$$\dot{h}_{ab} = \tilde{\mathcal{L}}_{\vec{t}} h_{ab} = 2NK_{ab} + 2D_{(a} N_{b)} \tag{70}$$

## 8 Canonical Variables of Gravitational Field

The derivation in this section follows the standard ADM decomposition of the Einstein–Hilbert action, up to boundary terms [5, 3, 4].

To write down the Hamiltonian mechanics of the gravitational field, we should start from the action and Lagrangian density. Here we consider the gravitational field in the vacuum.

$$S_{EH} = \int R\sqrt{-g} d^4x = \int g^{ab}R_{ab}\sqrt{-g} d^4x \quad (71)$$

Through the principle of least action, we obtained the equations of motion for the gravitational field in the vacuum.

$$\delta S_{EH} = \int \left( R_{ab} - \frac{1}{2}g_{ab}R \right) \delta g^{ab} \sqrt{-g} d^4x \quad (72)$$

$$\delta S_{EH} = 0 \Rightarrow R_{ab} - \frac{1}{2}g_{ab}R = 0 \quad (73)$$

We derive the equation of motion by varying the action with respect to the metric field, which means that the metric field is the generalized coordinate among the canonical variables. With this understanding, our goal is to rewrite the metric field as a generalized coordinate in the form of a 3+1 decomposition. According to Eq. (41), the metric can be written as:

$$g^{ab} = h^{ab} - n^a n^b = h^{ab} - N^{-2}(t^a - N^a)(t^b - N^b) \quad (74)$$

$$\Rightarrow g^{ab}|_{\Sigma_t} \text{ determined by } (N, N_a, h_{ab})|_{\Sigma_t} \quad (75)$$

According to the results of the metric rewriting, under the 3+1 decomposition, there are three kinds of generalized coordinates of the gravitational field: the lapse function  $N$ , the dual shift vector field  $N_a$ , and the spatial metric field  $h_{ab}$ .

The next step is to rewrite the Lagrangian density using the 3+1 decomposition method. We have already obtained the 3+1 version of the Ricci scalar in Eq. (56), so we can write it directly:

$$\begin{aligned} \mathcal{L} &= R\sqrt{-g} \\ &= ({}^3R + K^2 - K_{cb}K^{cb} - 2n^e n^g R_{eg})\sqrt{-g} \end{aligned} \quad (76)$$

The last term,  $n^e R_{eg} n^g$ , requires further calculation to determine its properties.

$$\begin{aligned} n^e R_{eg} n^g &= n^e R_{ecg} n^g \\ &= -n^e (\nabla_e \nabla_c - \nabla_c \nabla_e) n^c \\ &= -n^e \nabla_e (\nabla_c n^c) + n^e \nabla_c (\nabla_e n^c) \\ &= -\nabla_e (n^e \nabla_c n^c) + (\nabla_e n^e) \nabla_c n^c + \nabla_c (n^e \nabla_e n^c) - (\nabla_c n^e) \nabla_e n^c \\ &= -\nabla_e (n^e \nabla_c n^c) + K^2 + \nabla_c (n^e \nabla_e n^c) - K_{ce} K^{ce} \end{aligned} \quad (77)$$

After rewriting the Lagrangian density, the following results were obtained.

$$\begin{aligned} \mathcal{L} &= R\sqrt{-g} \\ &= ({}^3R + K^2 - K_{cb}K^{cb} - 2K^2 + 2K_{ce}K^{ce} + 2\nabla_e (n^e \nabla_c n^c) - 2\nabla_c (n^e \nabla_e n^c))\sqrt{-g} \\ &= ({}^3R - K^2 + K_{cb}K^{cb})\sqrt{-g} + 2(\nabla_e (n^e \nabla_c n^c) - \nabla_c (n^e \nabla_e n^c))\sqrt{-g} \end{aligned} \quad (78)$$

Consider integrating the Lagrangian density over spacetime; the last two terms in Eq. (78) can be regarded as boundary terms using the divergence theorem.

$$\mathcal{L}_{\text{boundary}} = 2(\nabla_e(n^e \nabla_c n^c) - \nabla_c(n^e \nabla_e n^c))\sqrt{-g} \quad (79)$$

Boundary terms can be ignored under certain assumptions, such as assuming the universe is a compact manifold without boundaries, or that spacetime is asymptotically flat. However, a more stringent condition is that the Gibbons–Hawking–York boundary term must be introduced to eliminate boundary terms truly. The following discussion directly assumes that boundary terms are eliminable and ignores them.

The next goal is to rewrite  $\sqrt{-g}$  as a 3+1 decomposition version. Since  $\sqrt{-g}$  comes from the volume form  $\epsilon$ , its definition is as follows:

$$\begin{cases} \epsilon_{abcd} := \sqrt{-g} (dt)_a \wedge (dx^1)_b \wedge (dx^2)_c \wedge (dx^3)_d \\ \epsilon := \sqrt{-g} dt \wedge dx^1 \wedge dx^2 \wedge dx^3 \end{cases} \quad (80)$$

If we define a spatial volume form by contracting the time direction of the volume form with the normal vector  $n^a$  of the spatial hypersurface as follows:

$${}^3\epsilon := n^a \epsilon_{abcd} \quad (81)$$

We found that this definition is actually flawed because, according to Eq. (39),  $(dx^i)_a$  is not necessarily a spatial tensor. Therefore,  ${}^3\epsilon$  defined in this way may not satisfy the definition of a spatial tensor. Furthermore,  ${}^3\epsilon$  defined in this way will change over time, that is:

$$\tilde{\mathcal{L}}_t {}^3\epsilon \neq 0 \quad (82)$$

To ensure that the newly defined spatial volume form does not change over time, we modify the definition as follows to remove the effects caused by time-related functions in the metric.

$$\begin{cases} e_{abcd} := (dt)_a \wedge (dx^1)_b \wedge (dx^2)_c \wedge (dx^3)_d \\ \mathbf{e} := dt \wedge dx^1 \wedge dx^2 \wedge dx^3 \\ t^a e_{abcd} = dx^1 \wedge dx^2 \wedge dx^3 \end{cases} \quad (83)$$

Furthermore, to ensure that it satisfies the definition of a spatial tensor, we project it using a projection matrix and finally define the spatial volume form  ${}^3\mathbf{e}$  as follows:

$${}^3\mathbf{e} := h_b^m h_c^n h_d^l t^a e_{amnl} \equiv (t^a e_{abcd})^\sim \quad (84)$$

In this way, the integral of spacetime can be rewritten using the time-invariant spatial volume form  ${}^3\mathbf{e}$  as follows:

$$\tilde{\mathcal{L}}_t {}^3\mathbf{e} = 0 \Rightarrow \int d^4x = \int dt \int_{\Sigma_t} {}^3\mathbf{e} \quad (85)$$

Next, we can obtain the description of  $\sqrt{-g}$  under 3+1 decomposition using the following method.

$$\begin{aligned}
t^a \sqrt{-g} e_{abcd} &= t^a \epsilon_{abcd} = N n^a \epsilon_{abcd} + N^a \epsilon_{abcd} = N n^a \epsilon_{abcd} \\
\Rightarrow {}^3e &= (t^a e_{abcd})^\sim = \left( \frac{N}{\sqrt{-g}} {}^3\epsilon \right)^\sim = \left( \frac{N\sqrt{h}}{\sqrt{-g}} {}^3e \right) \\
\Rightarrow \sqrt{-g} &= N\sqrt{h}
\end{aligned} \tag{86}$$

Combining Eq. (70), Eq. (78), and Eq. (86), and ignoring boundary terms, the Lagrangian density can be written as:

$$\begin{cases} \mathcal{L} = ({}^3R - K^2 + K_{cb}K^{cb})N\sqrt{h} \\ K_{ab} = \frac{1}{2}N^{-1}(\dot{h}_{ab} - 2D_{(a}N_{b)}) \end{cases} \tag{87}$$

Since we know from Eq. (75) that there are three generalized coordinates  $(N, N_a, h_{ab})$ . When deriving the canonical momentum from the Lagrangian density, we can immediately obtain two types of primary constraints, one associated with the lapse and the other a vector-valued constraint associated with the shift.

$$\begin{cases} \pi_N \equiv \partial\mathcal{L}/\partial\dot{N} = 0 \\ \pi_{N_a} \equiv \partial\mathcal{L}/\partial\dot{N}_a = 0 \end{cases} \tag{88}$$

The only non-zero canonical momentum is the canonical momentum  $\pi^{ab}$  corresponding to the spatial metric field  $h_{ab}$  as a generalized coordinate.

$$\pi^{ab} \equiv \frac{\partial\mathcal{L}}{\partial\dot{h}_{ab}} = 2\sqrt{h}N \left( K^{cd} \frac{\partial K_{cd}}{\partial\dot{h}_{ab}} - K \frac{\partial K}{\partial\dot{h}_{ab}} \right) \tag{89}$$

Where

$$\frac{\partial K_{cd}}{\partial\dot{h}_{ab}} = \frac{1}{2N} \frac{\partial\dot{h}_{cd}}{\partial\dot{h}_{ab}} = \frac{1}{2N} \delta_c^{(a} \delta_d^{b)} \tag{90}$$

and

$$\frac{\partial K}{\partial\dot{h}_{ab}} = \frac{\partial(h^{cd}K_{cd})}{\partial\dot{h}_{ab}} = h^{cd} \frac{1}{2N} \delta_c^{(a} \delta_d^{b)} = \frac{1}{2N} h^{ab} \tag{91}$$

After the above calculations and simplifications, the canonical momentum  $\pi^{ab}$  of the spatial metric field is:

$$\pi^{ab} = \sqrt{h}(K^{ab} - Kh^{ab}) \tag{92}$$

According to the Dirac-Bergmann algorithm mentioned at the beginning, the derivative of the generalized coordinate with respect to time, excluding the primary constraints, can be solved by the canonical momentum and the generalized coordinate. The following further verifies this matter.

$$\pi = h_{ab}\pi^{ab} = \sqrt{h}(h_{ab}K^{ab} - Kh_{ab}h^{ab}) = \sqrt{h}(K - 3K) = -2\sqrt{h}K \tag{93}$$

$$\begin{aligned}
& \begin{cases} K = -\pi/(2\sqrt{h}) \\ \pi^{ab} = \sqrt{h}(K^{ab} - Kh^{ab}) \end{cases} \\
& \Rightarrow K_{ab} = \frac{1}{\sqrt{h}} \left( \pi_{ab} - \frac{1}{2} \pi h_{ab} \right)
\end{aligned} \tag{94}$$

We can see that  $\dot{h}_{ab}$  is indeed solved in terms of canonical momentum  $\pi^{ab}$  and generalized coordinates  $(N, N_a, h_{ab})$

$$\dot{h}_{ab} = 2NK_{ab} + 2D_{(a}N_{b)} = 2N \frac{1}{\sqrt{h}} \left( \pi_{ab} - \frac{1}{2} \pi h_{ab} \right) + 2D_{(a}N_{b)} \tag{95}$$

## 9 Hamiltonian Formulation of Gravitational Field

Now that we understand all the canonical variables  $(N, N_a, h_{ab}; \pi^{ab})$  required for Hamiltonian mechanics of gravitational fields, we can rewrite the Lagrangian density using canonical variables as follows:

$$\mathcal{L} = ({}^3R - K^2 + K_{cb}K^{cb})N\sqrt{h} = N\sqrt{h} \left[ {}^3R + \frac{1}{h} \left( \pi^{ab}\pi_{ab} - \frac{1}{2}\pi^2 \right) \right] \tag{96}$$

Now, through the Legendre transformation, we obtain the Hamiltonian density.

$$\begin{aligned}
\mathcal{H} &= \pi^{ab}\dot{h}_{ab} - \mathcal{L} \\
&= \frac{2N}{\sqrt{h}} \left( \pi^{ab}\pi_{ab} - \frac{1}{2}\pi\pi^{ab}h_{ab} \right) + 2\pi^{ab}D_{(a}N_{b)} - \mathcal{L} \\
&= \frac{2N}{\sqrt{h}} \left( \pi^{ab}\pi_{ab} - \frac{1}{2}\pi^2 \right) + 2\pi^{ab}D_{(a}N_{b)} - \mathcal{L} \\
&= N\sqrt{h} \left[ -{}^3R + \frac{1}{h} \left( \pi^{ab}\pi_{ab} - \frac{1}{2}\pi^2 \right) \right] + 2\pi^{ab}D_{(a}N_{b)}
\end{aligned} \tag{97}$$

$$\begin{aligned}
H &= \int_{\Sigma_t} {}^3e \mathcal{H} \\
&= \int_{\Sigma_t} {}^3e N \sqrt{h} \left[ -{}^3R + \frac{1}{h} \left( \pi^{ab}\pi_{ab} - \frac{1}{2}\pi^2 \right) \right] \\
&\quad - 2 \int_{\Sigma_t} {}^3e \sqrt{h} N_b D_a \left( \frac{\pi^{ab}}{\sqrt{h}} \right) \\
&\quad + 2 \int_{\Sigma_t} {}^3e \sqrt{h} D_a \left( \frac{N_b \pi^{ab}}{\sqrt{h}} \right)
\end{aligned} \tag{98}$$

With the Hamiltonian density, and by utilizing Eq. (85) and neglecting boundary terms, we finally wrote down the Hamiltonian for the gravitational field under 3+1 decomposition.

$$\begin{aligned}
& H[N, N_b, h_{ab}; \pi^{ab}] \\
&= \int_{\Sigma_t} {}^3e N \sqrt{h} \left[ -{}^3R + \frac{1}{h} \left( \pi^{ab}\pi_{ab} - \frac{1}{2}\pi^2 \right) \right] - 2 \int_{\Sigma_t} {}^3e \sqrt{h} N_b D_a \left( \frac{\pi^{ab}}{\sqrt{h}} \right)
\end{aligned} \tag{99}$$

According to the Dirac-Bergmann algorithm, the gravitational field now has two primary constraints (Eq. (88)), each of which should satisfy the consistency condition in Eq. (17). However, since  $H$  is a functional, we also need to rewrite the canonical momentum  $\pi_N$  and  $\pi_{N_a}$ , which serve as primary constraints, as functionals. Let  $\pi_N$  and  $\pi_{N_a}$  each integrate over spacetime. We name the resulting functionals  $\Pi_N$  and  $\Pi_{N_a}$ .

$$\begin{cases} \Pi_N = \int_{\Sigma_t} {}^3e\sqrt{h}\pi_N \\ \Pi_{N_a} = \int_{\Sigma_t} {}^3e\sqrt{h}\pi_{N_a} \end{cases} \quad (100)$$

Thus, the consistency condition can be read as:

$$\begin{cases} \{\Pi_N, H\} + \{\Pi_N, \Pi_{N_a}\}\lambda_{N_a} \approx 0 \\ \{\Pi_{N_a}, H\} + \{\Pi_{N_a}, \Pi_N\}\lambda_N \approx 0 \end{cases} \quad (101)$$

From Eq. (100), we know that  $\{\Pi_N, \Pi_{N_a}\} \approx 0$ . However, according to Eq. (99),  $\{\Pi_N, H\} = \dot{\pi}_N$  and  $\{\Pi_{N_a}, H\} = \dot{\pi}_{N_a}$  may not be zero. Therefore, the secondary constraints are given as follows:

$$\begin{cases} (\delta H/\delta N) = \sqrt{h}[-{}^3R + \frac{1}{h}(\pi^{ab}\pi_{ab} - \frac{1}{2}\pi^2)] = 0 \\ (\delta H/\delta N_b) = -2\sqrt{h}D_a\left(\frac{\pi^{ab}}{\sqrt{h}}\right) = 0 \end{cases} \quad (102)$$

We denote the secondary constraint of the scalar property as  $C$ .

$$C \equiv -{}^3R + \frac{1}{h}\left(\pi^{ab}\pi_{ab} - \frac{1}{2}\pi^2\right) = 0 \quad (103)$$

We denote the secondary constraint of the vector property as  $C^b$ .

$$C^b \equiv -2D_a(h^{-1/2}\pi^{ab}) = 0 \quad (104)$$

These are the standard ADM Hamiltonian and momentum constraints [5, 3].

We can further verify that the Poisson brackets of all primary and secondary constraints are zero on the constraint surface. Therefore, according to Dirac's classification, the gravitational field system belongs to the first-class of constraint systems. This also means that the two primary constraints correspond to two free Lagrange multipliers (in this case, this means one scalar field and one vector field).

$$\begin{cases} \dot{N} = (\delta H/\delta \pi_N) + (\delta \Pi_N/\delta \pi_N)\lambda_N = \sqrt{h}\lambda_N \\ \dot{N}_a = (\delta H/\delta \pi_{N_a}) + (\delta \Pi_{N_a}/\delta \pi_{N_a})\lambda_{N_a} = \sqrt{h}\lambda_{N_a} \end{cases} \quad (105)$$

Now let's rewrite the Hamiltonian using secondary constraints. After combining Eq. (99), Eq. (103) and Eq. (104), the Hamiltonian reads

$$H[N, N_b, h_{ab}; \pi^{ab}] = \int_{\Sigma_t} {}^3e\sqrt{h}\left(NC + N^b C_b\right) \quad (106)$$

This formulation is also called the Arnowitt-Deser-Misner (ADM) formalism. Suppose we consider the Hamiltonian as the generator of evolution in Eq. (106). The scalar secondary

constraint generates a lapse function " $N$ " related to time selection, while the vector secondary constraint generates a dual shift vector field " $N_b$ " related to space selection.

It is time to discuss the implications of gauge degrees of freedom in the context of Hamiltonian mechanics. Gauge degrees of freedom are generally defined as the degrees of freedom derived from an infinite-dimensional group when the action is invariant under its influence. This invariance imposes constraints on the canonical variables, which form orbits on the constraint surface. Points on the same orbit can sometimes be considered equivalent descriptions of the same physical state, while at other times they should be viewed as different stages in the evolution of the physical system. The discussion of gauge only aligns with gauge theory in the first case. The second case concerns the degree of arbitrariness in time as a parameter describing evolution; this process is usually called parameterization.

In physics, interesting theories typically have constraints associated with gauge that are linear functions of canonical momentum, while those associated with parameterization are often quadratic functions of canonical momentum (Eq. (103)). In field systems, electrodynamics and Yang-Mills theory are traditional gauge theories, in which the constraints are linear and homogeneous with respect to the canonical momentum. A typical example of parameterization theory is general relativity, where constraints associated with the arbitrariness of time choice are quadratic in canonical momentum (Eq. (103)), and there are also linear diffeomorphism constraints related to the invariance under spatial diffeomorphisms (Eq. (104)), which act as gauge groups.

## 10 Ashtekar's New Variables

The reformulation of canonical general relativity in terms of complex connection variables follows Ashtekar's new Hamiltonian formulation [6, 7, 8].

From the previous sections, we have learned that in the vacuum, the gravitational field's true dynamical canonical variables are the spatial metric field  $h_{ab}$  and its conjugate momentum  $\pi^{ab}$ . Under this formulation, the constraints possess a non-polynomial structure (Eq. (103)). According to the quantization procedure, once we promote the constraint equations to operator equations, they become difficult to solve. However, in principle, this difficulty could be overcome by performing an appropriately chosen canonical transformation on the canonical variables.

Traditional canonical variables  $(h_{ab}, \pi^{ab})$  are a pair of tensor fields defined on the three-dimensional manifold  $\Sigma_t$ . Let  $\{e_i^a \mid i = 1, 2, 3\}$  denote a local frame field (vielbein) on  $\Sigma_t$ , and  $\{e_a^i\}$  its dual frame field (dual vielbein). These frame fields are required to be orthonormal with respect to the spatial metric  $h_{ab}$ .

$$\begin{cases} h_{ab} = \delta_{ij} e_a^i e_b^j \\ e_a^i e_j^a = \delta_j^i \\ e_a^i e_i^b = \delta_a^b \end{cases} \quad (107)$$

However, requiring orthonormality with respect to the spatial metric  $h_{ab}$  does not uniquely determine  $\{e_i^a\}$ . Rotations and reflections (i.e., an overall sign flip) of  $\{e_i^a\}$  by elements of  $O(3)$  do not affect their orthonormality under the metric  $h_{ab}$ . In the formulation with the new variables,  $\{e_i^a\}$  will replace  $h_{ab}$  as the canonical variables. In accordance with the structure of the Lagrangian density, we first define the following objects in the same local frame.

$$e_j^i = e_a^i \left( \frac{\partial}{\partial x^j} \right)^a \quad (108)$$

$$\det(e) \equiv \det(e_j^i) \quad (109)$$

$$h = [\det(e)]^2 \quad (110)$$

$$\epsilon \equiv \text{sgn}(\det(e)) \quad (111)$$

Now we define a densitized frame field in three dimensions as follows:

$$E_i^a := \det(e) e_i^a \quad (112)$$

Since  $\det(e)$  can cancel out the negative sign during the inversion of  $\{e_i^a\}$ , the degree of freedom of  $E_i^a$  is restricted to a rotation of  $SO(3)$ . Given that  $K_{ab}$  is the extrinsic curvature of  $\Sigma_t$  in Eq. (43), we define

$$K_a^i := \epsilon K_{ab} e_j^b \delta^{ij} \quad (113)$$

However, using  $(E_i^a, K_a^i)$  as the new canonical variables cannot simplify the constraint  $C = 0$  in Eq. (103). Ashtekar discovered that, in order to achieve a simplification of the constraints, one must modify the connection compatible with the spatial metric field  $h_{ab}$ . Here we start from the compatibility of the spatial derivative:

$$\begin{aligned} D_a h_{bc} &= 0 \\ \Rightarrow D_a (\delta_{ij} e_b^i e_c^j) &= 0 \\ \Rightarrow D_a e_b^i &= 0 \\ \Rightarrow \partial_a e_b^i + \omega_a^i{}_j e_b^j - \Gamma_{ab}^c e_c^i &= 0 \end{aligned} \quad (114)$$

Using the orthonormality of the frame field, we have the connection one-form (spin connection):

$$\begin{aligned} \partial_a e_b^i + \omega_a^i{}_j e_b^j - \Gamma_{ab}^c e_c^i &= 0 \\ \Rightarrow \omega_a^i{}_j e_b^j &= -(\partial_a e_b^i - \Gamma_{ab}^c e_c^i) \\ \Rightarrow \omega_a^i{}_j e_b^j e_k^b &= -e_k^b (\partial_a e_b^i - \Gamma_{ab}^c e_c^i) \\ \Rightarrow \omega_a^i{}_k &= -e_k^b \nabla_a e_b^i \\ \Rightarrow \omega_a^i{}_k &= e_b^i \nabla_a e_k^b \end{aligned} \quad (115)$$

Before returning to the three-dimensional spatial manifold  $\Sigma_t$ , it is useful to give a simple example showing explicitly how the spin connection is computed from a vielbein. Consider a two-sphere  $S^2$  with radius  $R$ . Its metric is

$$g_{ab} = R^2 (d\theta)_a (d\theta)_b + R^2 \sin^2 \theta (d\phi)_a (d\phi)_b.$$

Here  $a, b, \dots$  are abstract tensor indices on  $S^2$ , whereas  $i, j, \dots = 1, 2$  are internal frame indices. The dual vielbein  $e^i{}_a$  such that

$$g_{ab} = \delta_{ij} e^i{}_a e^j{}_b.$$

A natural choice is

$$e^1{}_a = R(d\theta)_a, \quad e^2{}_a = R \sin \theta (d\phi)_a.$$

The corresponding vielbein  $e^a_i$  satisfies

$$e^i_a e^a_j = \delta^i_j, \quad e^i_a e^b_i = \delta^b_a,$$

and is given by

$$e^a_1 = \frac{1}{R} \left( \frac{\partial}{\partial \theta} \right)^a, \quad e^a_2 = \frac{1}{R \sin \theta} \left( \frac{\partial}{\partial \phi} \right)^a.$$

The spin connection is determined by requiring the vielbein to be covariantly constant. In Cartan's language, this is equivalently expressed by the first structure equation. Here is a simple derivation. First, take the exterior derivative of the dual vielbein:

$$d_c e^i_a = \partial_\nu e^i_\mu (dx^\nu)_c \wedge (dx^\mu)_a \quad (116)$$

And use Eq. (114) to substitute the  $\partial_\nu e^i_\mu$ , one gets

$$\begin{aligned} d_c e^i_a &= (\Gamma^d_{\nu\mu} e^i_d - \omega^i_{\nu j} e^j_\mu) (dx^\nu)_c \wedge (dx^\mu)_a \\ \Rightarrow d_c e^i_a + \omega^i_{c j} e^j_a &= \left( \Gamma^d_{\nu\mu} (dx^\nu)_c \wedge (dx^\mu)_a \right) e^i_d \end{aligned} \quad (117)$$

If torsion vanishes, that is  $\Gamma^d_{\nu\mu} = \Gamma^d_{\mu\nu}$ , then

$$d_c e^i_a + \omega^i_{c j} e^j_a = 0 \quad (118)$$

Metric compatibility implies that the spin connection is antisymmetric in its internal indices:

$$\omega^i_{a j} = -\omega^j_{a i} \quad (119)$$

Therefore, in two dimensions the spin connection is  $\mathfrak{so}(2)$ -valued and has only one independent component:

$$\omega^1_{a 2} = -\omega^2_{a 1}$$

We now compute the connection explicitly. From

$$e^1 = R d\theta, \quad e^2 = R \sin \theta d\phi,$$

we obtain

$$de^1 = 0,$$

and

$$de^2 = R \cos \theta d\theta \wedge d\phi.$$

Since

$$e^1 \wedge e^2 = R^2 \sin \theta d\theta \wedge d\phi,$$

we can rewrite  $de^2$  as

$$de^2 = \frac{\cot \theta}{R} e^1 \wedge e^2.$$

The first Cartan structure equation gives two equations:

$$de^1 + \omega^1_{2} \wedge e^2 = 0,$$

and

$$de^2 + \omega^2_1 \wedge e^1 = 0.$$

The first equation is automatically satisfied if  $\omega^1_2$  is proportional to  $e^2$ . Thus, let

$$\omega^2_1 = fe^2.$$

Substituting this into the second structure equation gives

$$\frac{\cot \theta}{R} e^1 \wedge e^2 + fe^2 \wedge e^1 = 0.$$

Using

$$e^2 \wedge e^1 = -e^1 \wedge e^2,$$

we find

$$f = \frac{\cot \theta}{R}.$$

Therefore,

$$\omega^2_1 = \frac{\cot \theta}{R} e^2 = \cos \theta d\phi,$$

and hence

$$\omega^1_2 = -\cos \theta d\phi.$$

Equivalently, the spin connection can be written as

$$\omega^2_{a1} = \cos \theta (d\phi)_a, \quad \omega^1_{a2} = -\cos \theta (d\phi)_a.$$

In matrix form,

$$\{\omega^i_{aj}\} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \cos \theta (d\phi)_a.$$

This example makes the geometric meaning of the spin connection explicit. The vielbein  $e^i_a$  gives an orthonormal frame at each point of the curved manifold, but these local orthonormal frames are not globally constant. The spin connection  $(\omega_a)^i_j$  records how the internal frame rotates as one moves on the manifold. For the two-sphere, the relevant internal rotation group is  $SO(2)$ , so the spin connection is an  $\mathfrak{so}(2)$ -valued one-form. Sometimes, the spin connection people refer to is a further lift of the generator via an algebraic isomorphism. For example, the  $\mathfrak{so}(2)$  algebra corresponds to the  $\mathfrak{spin}(2)$  algebra, which is also the  $\mathfrak{u}(1)$  algebra.

In the Ashtekar formulation, the same idea is applied to the three-dimensional spatial manifold  $\Sigma_t$ . The spatial metric  $h_{ab}$  is written in terms of a triad  $e^i_a$ ,

$$h_{ab} = \delta_{ij} e^i_a e^j_b,$$

Since the densitized frame  $E^a_i = \det(e) e^a_i$  possesses an  $SO(3)$  rotational degree of freedom, in Cartan's terminology  $E^a_i$  functions as a moving frame. Describing the variation of this moving frame in terms of the frame itself leads to Cartan's connection one-form, that is, the spin connection. Moreover, because the motion of the frame is governed by  $SO(3)$  rotations, the connection one-form is an  $\mathfrak{so}(3)$ -Lie-algebra-valued one-form. This means that the connection is simultaneously a one-form and an element of the Lie algebra.

Let the basis of the Lie algebra  $\mathfrak{so}(3)$  be denoted by  $\{a_i \mid a_i \in \mathfrak{so}(3), i = 1, 2, 3\}$ .

$$a_1 \equiv \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad a_2 \equiv \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad a_3 \equiv \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \quad (120)$$

These matrices satisfy

$$[a_i, a_j] = \epsilon_{ij}^k a_k \quad (121)$$

We then write the  $SO(3)$  connection one-form as:

$$(\Gamma_b)^i_j = e_c^i \nabla_b e_j^c = (\Gamma_b^k a_k)^i_j = -\Gamma_b^k \epsilon_k^i_j \quad (122)$$

Ashtekar defined a new connection as follows:

$$A_a^j := \Gamma_a^j - iK_a^j, \quad j = 1, 2, 3 \quad (123)$$

The new connection implies a new derivative operator, which we denote by  $\mathcal{D}_a$  for the connection  $A_a^j$ . This new derivative corresponds to a new curvature tensor, or equivalently, by Cartan's curvature formula, the new connection gives rise to a curvature two-form. This curvature two-form is likewise a complex  $\mathfrak{so}(3)$ -Lie-algebra-valued two-form, which we denote by  $F_{ab}^i$ .

$$\begin{aligned} F_{ab} &= F_{ab}^i a_i = d_a(A_b^i a_i) + \frac{1}{2} \llbracket A_a^i a_i, A_b^j a_j \rrbracket \\ &= d_a(A_b^i a_i) + \frac{1}{2} [a_i, a_j] A_a^i \wedge A_b^j \\ &= \left( 2\partial_{[a} A_{b]}^i + \epsilon^i_{jk} A_{[a}^j A_{b]}^k \right) a_i \end{aligned} \quad (124)$$

### 10.1 Derivation of the Ashtekar Constraints

We now derive explicitly how the ADM constraints are rewritten in terms of Ashtekar's new variables, following the standard connection-dynamical formulation of canonical general relativity [7, 8]. The essential point is that the curvature of the complex connection

$$A_a^i := \Gamma_a^i - iK_a^i$$

contains both the intrinsic curvature of the spatial hypersurface and the extrinsic-curvature terms appearing in the ADM Hamiltonian constraint.

First, recall that the densitized triad is defined by

$$E_i^a := \det(e) e_i^a.$$

Since

$$h = [\det(e)]^2,$$

we have the basic identity

$$E_i^a E_j^b \delta^{ij} = [\det(e)]^2 e_i^a e_j^b \delta^{ij} = h h^{ab} \quad (125)$$

Therefore, the densitized triad contains the same metric information as  $h^{ab}$ , but with an additional internal  $SO(3)$  frame index.

The triadic form of the extrinsic curvature is

$$K_a^i := \epsilon K_{ab} e_j^b \delta^{ij}, \quad \epsilon := \text{sgn}(\det e).$$

For simplicity of notation, in the following derivation, we absorb the orientation sign  $\epsilon$  into the definition of the triad and write

$$K_a^i E_i^b = \sqrt{h} K_{ac} e^c_j \delta^{ji} e^b_i = \sqrt{h} K_{ac} \delta^{cb} = \sqrt{h} K_a^b \quad (126)$$

Contracting the two spatial indices gives

$$K_a^i E_i^a = \sqrt{h} K \quad (127)$$

Using the ADM momentum (Eq. (92))

$$\pi^{ab} = \sqrt{h}(K^{ab} - K h^{ab}),$$

we obtain

$$\pi^b_a = \sqrt{h}(K^b_a - K \delta^b_a) = K_a^i E_i^b - \delta_a^b K_c^i E_i^c \quad (128)$$

This formula is the bridge between the ADM momentum variable  $\pi^{ab}$  and the triad variables  $(K_a^i, E_i^a)$ .

### 10.1.1 The Gauss Constraint

Replacing the spatial metric  $h_{ab}$  by a triad introduces an internal rotational redundancy. Indeed, under a local  $SO(3)$  rotation

$$e_a^i \mapsto R^i_j e_a^j, \quad R^i_j \in SO(3),$$

the spatial metric is unchanged:

$$h_{ab} = \delta_{ij} e_a^i e_b^j \mapsto \delta_{ij} R^i_k R^j_l e_a^k e_b^l = \delta_{kl} e_a^k e_b^l = h_{ab}.$$

Thus, the triad description contains a gauge redundancy which is absent in the metric description. The corresponding constraint is the Gauss constraint [7, 8].

To avoid confusion with the spatial covariant derivative  $D_a$  compatible with  $h_{ab}$ , we denote the gauge-covariant derivative defined by the Ashtekar connection  $A_a^i$  by  $\mathcal{D}_a$ . Acting on an internal vector  $v_i$ , note that the quantities  $A_a^i$  are the coefficients of the  $\mathfrak{so}(3)$  basis in Eq (120), it is

$$\mathcal{D}_a v_i := \partial_a v_i - (A_a^j a_j)^k_i v_k = \partial_a v_i + \epsilon_{ij}^k A_a^j v_k \quad (129)$$

The Gauss constraint is

$$\mathcal{G}_i := \mathcal{D}_a E_i^a = 0 \quad (130)$$

Since

$$A_a^i = \Gamma_a^i - i K_a^i$$

We have

$$\mathcal{D}_a E_i^a = \partial_a E_i^a + \epsilon_{ij}^k (\Gamma_a^j - i K_a^j) E_k^a = D_a^\Gamma E_i^a - i \epsilon_{ij}^k K_a^j E_k^a \quad (131)$$

The spin connection  $\Gamma_a^i$  is compatible with the triad, so

$$D_a^\Gamma E_i^a = 0.$$

Therefore

$$\mathcal{G}_i = -i\epsilon_{ij}{}^k K_a^j E_k^a. \quad (132)$$

Hence

$$\mathcal{G}_i = 0 \quad \Longleftrightarrow \quad \epsilon_{ij}{}^k K_a^j E_k^a = \sqrt{h}\epsilon_i{}^{jk} K_{ab} e_j^b e_k^a = \sqrt{h}\epsilon_i{}^{jk} K_{kj} = \sqrt{h}\epsilon_i{}^{[jk]} K_{(kj)} = 0 \quad (133)$$

This is precisely the internal-index form of the symmetry of the extrinsic curvature:

$$K_{ab} = K_{ba} = K_{(ab)}$$

Therefore the Gauss constraint is not an additional physical restriction on the ADM phase space. It is the constraint generated by the internal  $SO(3)$  redundancy introduced when  $h_{ab}$  is replaced by a triad.

### 10.1.2 Curvature of the Ashtekar Connection

The curvature of  $A_a^i$  shown in Eq. (124) is

$$F_{ab}^i = 2\partial_{[a} A_{b]}^i + \epsilon^i{}_{jk} A_{[a}^j A_{b]}^k$$

Substituting

$$A_a^i = \Gamma_a^i - iK_a^i$$

gives

$$\begin{aligned} F_{ab}^i(A) &= 2\partial_{[a}\Gamma_{b]}^i + \epsilon^i{}_{jk}\Gamma_{[a}^j\Gamma_{b]}^k - 2i\partial_{[a}K_{b]}^i - i\epsilon^i{}_{jk}\Gamma_{[a}^jK_{b]}^k - i\epsilon^i{}_{jk}K_{[a}^j\Gamma_{b]}^k - \epsilon^i{}_{jk}K_{[a}^jK_{b]}^k \\ &= R_{ab}^i(\Gamma) - 2iD_{[a}K_{b]}^i - \epsilon^i{}_{jk}K_{[a}^jK_{b]}^k \end{aligned} \quad (134)$$

Here

$$R_{ab}^i(\Gamma) := 2\partial_{[a}\Gamma_{b]}^i + \epsilon^i{}_{jk}\Gamma_{[a}^j\Gamma_{b]}^k \quad (135)$$

is the curvature of the triad-compatible spin connection. Equation (134) is the key identity behind the simplification of the gravitational constraints [7, 8]: The first term gives the intrinsic curvature of  $\Sigma_t$ , while the last term gives precisely the quadratic extrinsic curvature contribution.

### 10.1.3 The Vector Constraint

The ADM momentum constraint obtained earlier is

$$C_a = -2D_b(h^{-1/2}\pi_a^b) = 0.$$

From Eq. (92), using

$$h^{-1/2}\pi_a^b = K_a^b - K\delta_a^b,$$

we obtain

$$C_a = -2D_b(K_a^b - K\delta_a^b) = -2(D_bK_a^b - D_aK) \quad (136)$$

Thus, the ADM momentum constraint is equivalent to

$$D_bK_a^b - D_aK = 0 \quad (137)$$

Now consider the Ashtekar vector constraint candidate

$$\mathcal{V}_a := F_{ab}^i E_i^b \quad (138)$$

Using the curvature decomposition from Eq. (134),

$$F_{ab}^i = R_{ab}^i(\Gamma) - 2iD_{[a}^\Gamma K_{b]}^i - \epsilon^i{}_{jk} K_{[a}^j K_{b]}^k,$$

we find

$$\mathcal{V}_a = R_{ab}^i(\Gamma) E_i^b - iE_i^b (D_a^\Gamma K_b^i - D_b^\Gamma K_a^i) - \epsilon^i{}_{jk} K_{[a}^j K_{b]}^k E_i^b \quad (139)$$

The first term vanishes by the first Bianchi identity for the spatial Riemann tensor:

$$R_{ab}^i(\Gamma) E_i^b = -\frac{1}{2} \epsilon_j{}^{ki} ({}^3R)^j{}_{kab} E_i^b = -\left[ ({}^3R)^1{}_{2ab} E_3^b + ({}^3R)^3{}_{1ab} E_2^b + ({}^3R)^2{}_{3ab} E_1^b \right] = 0 \quad (140)$$

The last term vanishes on the Gauss constraint surface, because the Gauss constraint is equivalent to the symmetry of  $K_{ab}$ .

$$\epsilon^i{}_{jk} K_{[a}^j K_{b]}^k E_i^b = \epsilon^i{}_{jk} K_{[a|c} e_n^c \delta^{jn} K_{|b]d} e_m^d \delta^{km} E_i^b = -\sqrt{h} \epsilon^{jik} K_{j[a} K_{i]k} = -\sqrt{h} \epsilon^{j[ik]} K_{j[a} K_{i]k} = 0 \quad (141)$$

Therefore, modulo the Gauss constraint,

$$\mathcal{V}_a \approx -iE_i^b (D_a^\Gamma K_b^i - D_b^\Gamma K_a^i) \quad (142)$$

Since  $D_a^\Gamma$  is compatible with the triad, we may convert internal indices back into spatial indices:

$$E_i^b D_a^\Gamma K_b^i = \sqrt{h} D_a^\Gamma (e_i^b K_b^i) = \sqrt{h} D_a K \quad (143)$$

and

$$E_i^b D_b^\Gamma K_a^i = \sqrt{h} D_b^\Gamma (e_i^b K_a^i) = \sqrt{h} D_b K_a^b \quad (144)$$

Therefore

$$\mathcal{V}_a \approx -i\sqrt{h} (D_a K - D_b K_a^b) = i\sqrt{h} (D_b K_a^b - D_a K) \quad (145)$$

Using

$$C_a = -2(D_b K_a^b - D_a K),$$

we obtain

$$\mathcal{V}_a \approx -\frac{i\sqrt{h}}{2} C_a \quad (146)$$

Hence the ADM momentum constraint  $C_a = 0$  is equivalent, modulo the Gauss constraint, to the Ashtekar vector constraint [7, 8]

$$\boxed{\mathcal{V}_a := F_{ab}^i E_i^b = 0} \quad (147)$$

#### 10.1.4 The Scalar Constraint

The ADM Hamiltonian constraint obtained earlier in Eq. (103) is

$$C \equiv -{}^3R + \frac{1}{h} \left( \pi^{ab} \pi_{ab} - \frac{1}{2} \pi^2 \right) = 0.$$

From Eq. (92) and Eq. (93), using

$$h^{-1/2} \pi^{ab} = K^{ab} - K h^{ab}$$

and

$$-2K\sqrt{h} = \pi$$

we obtain

$$\begin{aligned} C &= -{}^3R + (K^{ab} - K h^{ab})(K_{ab} - K h_{ab}) - \frac{1}{2h} [2K\sqrt{h}]^2 \\ &= -{}^3R + K^{ab} K_{ab} - 2K^2 + 3K^2 - 2K^2 \\ &= -{}^3R + K^{ab} K_{ab} - K^2 = 0 \end{aligned} \quad (148)$$

We now show that this is equivalent to the polynomial Ashtekar scalar constraint [6, 7, 8]

$$\mathcal{S} := \epsilon^{jk} {}_i E_j^a E_k^b F_{ab}^i = 0 \quad (149)$$

Using the curvature decomposition Eq. (134),

$$F_{ab}^i = R_{ab}^i(\Gamma) - 2i D_{[a}^\Gamma K_{b]}^i - \epsilon^i{}_{mn} K_{[a}^m K_{b]}^n,$$

we get

$$\mathcal{S} = \epsilon^{jk} {}_i E_j^a E_k^b R_{ab}^i(\Gamma) - 2i \epsilon^{jk} {}_i E_j^a E_k^b D_{[a}^\Gamma K_{b]}^i - \epsilon^{jk} {}_i E_j^a E_k^b \epsilon^i{}_{mn} K_{[a}^m K_{b]}^n \quad (150)$$

The first term gives the spatial Ricci scalar. With the sign convention used here for the  $SO(3)$  curvature,

$$\epsilon^{jk} {}_i E_j^a E_k^b R_{ab}^i(\Gamma) = -h ({}^3R) \quad (151)$$

The second term vanishes because it is the contraction of an antisymmetric spatial volume factor with the symmetric extrinsic curvature. Explicitly, after converting internal indices into spatial indices,

$$\epsilon^{jk} {}_i E_j^a E_k^b D_{[a}^\Gamma K_{b]}^i = h \epsilon^{abc} D_{[a} K_{b]c} = h \epsilon^{[ab]c} D_a K_{bc} = h \epsilon^{abc} D_a K_{bc} \quad (152)$$

Since

$$K_{bc} = K_{cb} = K_{(bc)},$$

we have

$$\epsilon^{a[bc]} D_a K_{(bc)} = 0$$

Thus, the derivative term does not contribute to the scalar constraint.

For the last term, use

$$\epsilon^{jk} {}_i \epsilon^i{}_{mn} = \delta_m^j \delta_n^k - \delta_n^j \delta_m^k.$$

Then

$$\begin{aligned}
& -\epsilon^{jk}{}_i E_j^a E_k^b \epsilon^i{}_{mn} K_{[a}^m K_{b]}^n \\
& = (K_{[a}^m E_m^b)(K_{b]}^n E_n^a) - (K_{[a}^m E_m^a)(K_{b]}^n E_n^b) \\
& = \frac{1}{2} \left[ (K_a^m E_m^b)(K_b^n E_n^a) - (K_b^m E_m^b)(K_a^n E_n^a) - (K_a^m E_m^a)(K_b^n E_n^b) + (K_b^m E_m^a)(K_a^n E_n^b) \right] \\
& = h \left[ (K_a^m E_m^b)(K_b^n E_n^a) h^{-1} - (K_a^m E_m^a)(K_b^n E_n^b) h^{-1} \right]
\end{aligned} \tag{153}$$

From Eq. (126) and Eq. (127), using

$$K_a^i E_i^a = \sqrt{h} K,$$

and

$$K_a^i E_i^b = \sqrt{h} K_a^b,$$

we obtain

$$-\epsilon^{jk}{}_i E_j^a E_k^b \epsilon^i{}_{mn} K_{[a}^m K_{b]}^n = h(K_{ab} K^{ab} - K^2) \tag{154}$$

Combining the three terms gives

$$\mathcal{S} = -h^3 R + h(K_{ab} K^{ab} - K^2).$$

Therefore

$$\mathcal{S} = h \left( -^3R + K_{ab} K^{ab} - K^2 \right) \tag{155}$$

That is,

$$\boxed{\mathcal{S} = hC.} \tag{156}$$

Hence the ADM Hamiltonian constraint  $C = 0$  is equivalent to the Ashtekar scalar constraint

$$\boxed{\mathcal{S} := \epsilon^{jk}{}_i E_j^a E_k^b F_{ab}^i = 0} \tag{157}$$

### 10.1.5 The Total Hamiltonian in Ashtekar Variables

The three constraints in Ashtekar variables are therefore

$$\boxed{\mathcal{G}_i := \mathcal{D}_a E_i^a = 0,}$$

$$\boxed{\mathcal{V}_a := F_{ab}^i E_i^b = 0,}$$

and

$$\boxed{\mathcal{S} := \epsilon^{jk}{}_i E_j^a E_k^b F_{ab}^i = 0.}$$

The first is the Gauss constraint associated with the internal  $SO(3)$  rotation freedom of the triad. The second is the vector constraint, equivalent to the ADM momentum constraint modulo the Gauss constraint. The third is the scalar constraint, equivalent to the ADM Hamiltonian constraint.

Since

$$\mathcal{S} = hC,$$

the ADM lapse term

$$\sqrt{h}NC$$

can be written as

$$\sqrt{h}NC = \frac{N}{\sqrt{h}}\mathcal{S}.$$

It is therefore natural to introduce the densitized lapse

$$\tilde{N} := \frac{N}{\sqrt{h}} \quad (158)$$

Similarly, from Eq. (146)

$$2i\mathcal{V}_a \approx \sqrt{h}C_a,$$

the numerical factor relating the ADM vector constraint to  $\mathcal{V}_a$  can be absorbed into the Lagrange multiplier multiplying the vector constraint.

$$\tilde{N}^a := 2iN^a \quad (159)$$

Therefore, the total Hamiltonian can be written in the Ashtekar form

$$H[A, E] = \int_{\Sigma_t} {}^3e \left( \Lambda^i \mathcal{G}_i + \tilde{N}^a \mathcal{V}_a + \tilde{N} \mathcal{S} \right) \quad (160)$$

Equivalently,

$$H[A, E] = \int_{\Sigma_t} {}^3e \left[ \Lambda^i \mathcal{D}_a E_i^a + \tilde{N}^a F_{ab}^i E_i^b + \tilde{N} \epsilon^{jk}{}_i E_j^a E_k^b F_{ab}^i \right] \quad (161)$$

This expression makes the main advantage of Ashtekar's variables explicit: After introducing the densitized lapse, all constraints become polynomial functions of the connection  $A_a^i$ , its curvature  $F_{ab}^i$ , and the densitized triad  $E_i^a$  [6, 7, 8].

Mathematically, the transition from the traditional variables  $(h_{ab}, \pi^{ab})$  to the new variables  $(A_a^i, E_i^a)$  is a canonical transformation in phase space. Physically, it represents a shift of focus from the evolution of the spatial metric field  $h_{ab}$  to the evolution of the connection  $A_a^i$ , namely **connection dynamics**. The advantage of this reformulation is that it makes the gravitational theory closely resemble an ordinary Yang–Mills gauge theory. The additional Gauss constraint  $\mathcal{D}_a E_i^a = 0$  is highly analogous to the Gauss constraint in Yang–Mills theory, thereby situating the problem of quantizing gravity within the same structural framework as the other fundamental interactions.

### 10.1.6 Reality Conditions

One final point should be emphasized. The connection

$$A_a^i = \Gamma_a^i - iK_a^i$$

is complex. Therefore, the Ashtekar phase space is a complexified phase space. To recover real Lorentzian general relativity, one must impose reality conditions [8]. A convenient form is

$$E_i^a E^{bi} = hh^{ab} \in \mathbb{R} \quad (162)$$

and

$$A_a^i + \overline{A_a^i} = 2\Gamma_a^i(E) \quad (163)$$

Equivalently, one may require

$$E_i^a \in \mathbb{R}, \quad K_a^i = \frac{i}{2} (A_a^i - \overline{A_a^i}) \in \mathbb{R} \quad (164)$$

Thus, the complex Ashtekar variables simplify the constraint structure, but the reality conditions are necessary to ensure that the resulting geometry corresponds to a real Lorentzian spacetime.

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